A Dynamic Model with Endogenous Retirement: Existence of Multiple Steady States[1]

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Abstract

This paper extends Matsuyama's (2006) 0-1 endogenous-retirement choice in the second period to a framework with a continuous-endogenous-retirement choice to study consumption-saving decision and capital accumulation in an overlapping-generations model. The existence of the steady state is shown, and the conditions for the existence of multiple steady states are provided for both Matsuyama's (2006) 0-1 endogenous-retirement choice and the continuous-endogenous-retirement choice models, respectively. Different from Matsuyama's (2006) 0 or 1 labor choice (a full-time employee or a full-time retiree) in the steady state, a partial retirement may be a stable equilibrium under the continuous endogenous retirement choice in the second period. Therefore, partial retirement may be the optimal choice, and the retirement choice depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level; if the initial capital stock is lower than a critical capital stock, then the individual will choose a relatively lower retirement level.

Key words: Overlapping-generations model; Endogenous retirement; Multiple steady states.

JEL Classification Numbers: D91, J13, J26

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Abstract

This paper extends Matsuyama's (2006) 0-1 endogenous-retirement choice in the second period to a framework with a continuous-endogenous-retirement choice to study consumption-saving decision and capital accumulation in an overlapping-generations model. The existence of the steady state is shown, and the conditions for the existence of multiple steady states are provided for both Matsuyama's (2006) 0-1 endogenous-retirement choice and the continuous-endogenous-retirement choice models, respectively. Different from Matsuyama's (2006) 0 or 1 labor choice (a full-time employee or a full-time retiree) in the steady state, a partial retirement may be a stable equilibrium under the continuous endogenous retirement choice in the second period. Therefore, partial retirement may be the optimal choice, and the retirement choice depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level; if the initial capital stock is lower than a critical capital stock, then the individual will choose a relatively lower retirement level.

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1. Introduction

The retirement decision is an interesting topic in macroeconomics. It is related to social security, aging, consumption-saving decision, education, productivity, employment, and many other critical issues in macroeconomics and labor economics. Its analytic framework relies on an assumption of the life cycle of an agent. Building on the seminal paper by Samuelson (1958) and on Diamond's (1965) classical overlapping-generations framework, Feldstein (1974) first analyzes the impact of social security on an individual's decision about retirement and saving by using an extended life-cycle model. He shows that social security depresses personal savings by about 30-50 percent. Hu (1979) next assumes that utility is time separable and concave in all variables to examine the consumption-saving decision in an overlapping-generations model with an endogenous retirement choice in the second period. He proves in his framework that people would choose to work for longer in the second period if the wages in the first period increased. In a long-run growth model with knowledge as a production factor, Romer (1986) treats the retirement decision exogenously and shows that: the economy can be growing over time and that the retirement actions of private agents can amplify the effects of small disturbances. Recently, Matsuyama (2006) has discussed the consumption-saving decisions in an overlapping-generations model with 0-1 endogenous-retirement choices in the second period. He shows that there may be multiple steady-state capital stocks and that the wage rate in the first period has a positive effect on the following generations. He also demonstrates that the economy will converge to the steady state slower than the traditional economy without an endogenous-retirement choice.

However, Feldstein (1974) discusses the possibility that the increased aggregate savings through induced retirement may be ended up in social security. He does not analyze the full general equilibrium effects of such induced retirement nor explores its growth implications. Hu's (1979) specialized assumptions on the utility function ensure the existence of a unique steady state. Matsuyama's (2006) model assumes that the endogenous retirement choice in the second period must be 0 or 1; in other words, the agent must decide whether to retire completely or work full time in the second period. This assumption is, of course, different from the reality.

This paper aims to extend Matsuyama's (2006) work by introducing an agent's continuous choices about retirement in the second period to examine the consumption-saving decision in an overlapping-generations model. Different from Matsuyama's (2006) 0 or 1 retirement choice model, in this paper, partial retirement may be the optimal choice. Furthermore, the conditions for the existence of multiple steady states in both Matsuyama's (2006) 0-1 endogenous retirement choice model and in the continuous retirement choice model are provided. It is found that the retirement choice depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level and the economy will converge to a equilibrium with relatively higher capital accumulation; if the initial capital stock is lower than a critical capital stock, then the individual will choose a relatively lower retirement level, and the economy will converge to an equilibrium with a relatively lower level of capital accumulation.

The rest of this paper is organized as follows. In Section 2, we present the basic model of this paper, which extends Matsuyama's (2006) 0-1 endogenous retirement choice model to a framework with a continuous endogenous retirement choice in the second period. Section 3 solves the model and presents the dynamics for the wages of an aggregate economy under the continuous endogenous retirement choice in the second period. The existence of the steady state and the conditions for the existence of multiple steady states are also provided in this section. Section 4 provides a numerical simulation of the model. Finally, we conclude our paper in Section 5.

2. Saving-Retirement Decision Problem

2.1 Individual choice

As in Matsuyama (2006), we assume that each person in an economy lives for two periods. He or she must work in the first period, but has a free choice in the second period: he or she can retire at any time in the second period, i.e., people have continuous retirement options. Note that this assumption is different from that of Matsuyama's (2006) model in which people have only two choices in the second period: work or retirement.

Furthermore, suppose that the size of each cohort is normalized to one, and people in the second period can only provide θ units of efficient labor if they supply one unit of physical labor. Here, θ is assumed to be a constant. The other assumptions are the same as the standard overlapping-generations model.

Let c_{t-1}^{y} stand for young people's consumption in the period $t-1$, c_t^o denote consumption by the old in the period t, s_t be savings in the period t, ω_t denote wages in the period t, and r_t be the interest rate. In addition, we assume that $e_t \in [0,1]$ stands for the choice of working time in the second period.

The consumer chooses consumption and labor supply to maximize utility

$$\max_{c_{t-1}^{y},\, c_t^o,\, e_t} U(c_{t-1}^{y},\ c_t^o,\ e_t)$$

subject to

$$c_t^o = (1+r_t)(\omega_{t-1} - c_{t-1}^y) + \omega_t \theta e_t$$

$$0 \le e_t \le 1$$

with a given ω_{t-1}.

For the convenience of our discussions, we make an additional assumption as follows.

Assumption 1: c_{t-1}^y and c_t^o are assumed to be weakly independent of e_t.

Under this additional assumption, we can write the utility function as the a specific form:

$$U(c_{t-1}^y, c_t^o, e_t) \equiv U(z_t, e_t),$$

where $z_t = Z(c_{t-1}^y,\ c_t^o) = (\frac{c_{t-1}^y}{1-\beta})^{1-\beta}(\frac{c_t^o}{\beta})^{\beta}$, and $\beta \in (0,\ 1]$ is a constant independent of e_t.

For simplicity, we specify the utility function $U(z_t, e_t)$ as

$$U(z_t, e_t) = z_t + \Lambda z_t^{\lambda}(1-e_t), \tag{1}$$

where $\Lambda > 0$ and $\lambda \in (1, \infty)$ are constants as in Matsuyama (2006).

To solve the optimization problem for the individual, we define the associated Lagrangian as

$$L = U(z_t, e_t) + \lambda_1 e_t + \lambda_2(1-e_t),$$

where λ_1 and λ_2 are the Lagrange multipliers associated with the labor choice constraint in the second period.

The first-order conditions necessary for optimization are

$$c_{t-1}^y = (1-\beta)(\omega_{t-1} + \tfrac{\omega_t}{1+r_t}\theta e_t), \tag{2a}$$

$$\omega_t\theta + \Lambda\lambda(1-e_t)\omega_t\theta z_t^{\lambda-1} - \Lambda z_t^{\lambda-1}\tfrac{c_t^o}{\beta} + (\lambda_1 - \lambda_2)(\tfrac{c_{t-1}^y}{1-\beta})^{\beta-1}(\tfrac{c_t^o}{\beta})^{1-\beta} = 0, \tag{2b}$$

$$\lambda_1 \ge 0, e_t \ge 0, \lambda_1 e_t = 0, \text{ and} \tag{2c}$$

$$\lambda_2 \ge 0, e_t \le 1, \lambda_2(1-e_t) = 0. \tag{2d}$$

For conditions (2c) and (2d), we consider the following four cases.

Case a: $\lambda_1 > 0,\ \lambda_2 > 0$.

In this case, we have $e_t = 0$ and $e_t = 1$, which leads to a contradiction. Therefore, we cannot obtain an optimal solution under this condition.

Case b: $\lambda_1 > 0,\ \lambda_2 = 0$.

Now, we have $e_t = 0$, and

$$c_{t-1}^y = (1-\beta)\omega_{t-1},$$

$$\lambda_1 = \Lambda R_t^{\beta\lambda}\omega_{t-1}^{\lambda} - \Lambda\lambda R_t^{\beta\lambda-1}\omega_{t-1}^{\lambda-1}\theta\omega_t - \theta\omega_t R_t^{\beta-1},$$

where we denote $R_t = 1+r_t$.

Case c: $\lambda_1 = 0,\ \lambda_2 > 0$.

Under this assumption, we have $e_t = 1$ and

$$c_{t-1}^y = (1-\beta)\omega_{t-1} + (1-\beta)\theta \tfrac{\omega_t}{R_t},$$

$$\lambda_2 = \theta\omega_t R_t^{\beta-1} - \Lambda R_t^{(\beta-1)\lambda}(R_t\omega_{t-1} + \theta\omega_t)^{\lambda}.$$

Case d: $\lambda_1 = 0, \lambda_2 = 0$.

In this case, from equations (2a) and (2b), we have

$$\omega_t\theta + \Lambda R_t^{\beta(\lambda-1)}\lambda(\omega_t\theta + R_t\omega_{t-1})(\omega_{t-1} + \frac{\omega_t}{R_t}\theta e_t)^{\lambda-1} - \Lambda(\lambda+1)R_t^{1-\beta+\beta\lambda}(\omega_{t-1} + \frac{\omega_t}{R_t}\theta e_t)^{\lambda} = 0. \quad (3)$$

Equation (3) determines an individual's working time choice e_t in the second period.

2.2. Firms

Suppose that output is produced by a standard constant-return-to-scale technology, $Y_t = F(K_t, L_t)$. To avoid complex mathematical computations, we use the Cobb-Douglas production function here, i.e.,

$$Y_t = AK_t^{\alpha}L_t^{1-\alpha},$$

where $A > 0$ and $0 < \alpha < 1$ are constants.

Let $k_t = \frac{K_t}{L_t}$ denote the capital-labor ratio, and from the firm's optimization problem, we have

$$R_t = 1 + r_t = f'(k_t), \quad \omega_t = f(k_t) - k_t f'(k_t).$$

Therefore, the gross interest rate equals the marginal productivity of capital, and the wage rate equals the marginal productivity of labor.

For simplicity, we assume that the depreciation factor $\delta = 1$.

2.3. Dynamics of the aggregate economy

Now, we focus on the aggregate economy of each cohort as a result of the optimal choices by individuals.

Each agent has the same choices about his/her retirement; therefore, the efficient labor at time t is $1 + \theta e_t$. Because there are no solutions in case a, we only consider the other three cases.

First, in case b, we have $e_t = 0$, and the individual's saving decision is

$$s_t = \beta\omega_{t-1}. \quad (4)$$

Hence, aggregate capital accumulation at time t can be derived as:

$$K_t = \beta\omega_{t-1}.$$

Therefore, the capital-labor ratio follows

$$k_t = K_t = \beta\omega_{t-1}. \quad (5)$$

Note that the determination of a firm's optimal choice conditions is

$$R_t = f'(k_t) = A\alpha k_t^{\alpha-1}, \quad \omega_t = f(k_t) - k_t f'(k_t) = (1-\alpha)Ak_t^{\alpha}. \tag{6}$$

Thus, we have

$$\frac{\omega_t}{R_t} = \frac{1-\alpha}{\alpha}\beta\omega_{t-1},$$

and the value of λ_1 can be determined as

$$\lambda_1 = [\alpha - \lambda(1-\alpha)\beta\theta]\Lambda A^{\beta\lambda}\alpha^{\beta\lambda-1}\beta^{\alpha\beta\lambda-\beta\lambda}\omega_{t-1}^{\alpha\beta\lambda-\beta\lambda+\lambda} - \theta A^{\beta}(1-\alpha)\alpha^{\beta-1}\beta^{\alpha\beta-\beta+1}\omega_{t-1}^{\alpha\beta-\beta+1}. \tag{7}$$

The nonnegative constraint of λ_1 requires

$$[\alpha - \lambda(1-\alpha)\beta\theta]\Lambda A^{\beta\lambda}\alpha^{\beta\lambda-1}\beta^{\alpha\beta\lambda-\beta\lambda}\omega_{t-1}^{\alpha\beta\lambda-\beta\lambda+\lambda} > \theta A^{\beta}(1-\alpha)\alpha^{\beta-1}\beta^{\alpha\beta-\beta+1}\omega_{t-1}^{\alpha\beta-\beta+1}. \tag{8}$$

Equation (8) implies that the individual will choose not to work in the second period if the efficiency of the physical labor provided by the old is too low (keeping the other parameters and ω_{t-1} unchanged). As a result, the old generation's consumption is offered by their savings in the first period. This becomes the standard overlapping-generations model without a retirement choice.

Secondly, in case c, we have $e_t = 1$, and capital accumulation follows

$$k_t = \frac{K_t}{1+\theta} = \frac{\omega_{t-1} - c_{t-1}^y}{1+\theta}.$$

Similar to the discussions in case b, we also have

$$k_t = M\omega_{t-1},$$

with $M = \frac{\beta}{1+\theta(\frac{1}{\alpha}-\frac{\beta}{\alpha}+\beta)}$. Then, the saving decision and the value of λ_2 can be derived as

$$s_t = \frac{(1+\theta)\beta}{1+\theta(\frac{1}{\alpha}-\frac{\beta}{\alpha}+\beta)}\omega_{t-1}, \tag{9}$$

$$\lambda_2 = \theta A^{\beta}\alpha^{\beta-1}(1-\alpha)M^{\alpha\beta-\beta+1}\omega_{t-1}^{\alpha\beta-\beta+1} - \Lambda(A\alpha)^{\beta\lambda}M^{(\alpha-1)\beta\lambda}(1+\theta\frac{1-\alpha}{\alpha}M)^{\lambda}\omega_{t-1}^{(\alpha\beta-\beta+1)\lambda}. \tag{10}$$

The non-negativity of λ_2 requires

$$\theta A^{\beta}\alpha^{\beta-1}(1-\alpha)M^{\alpha\beta-\beta+1}\omega_{t-1}^{\alpha\beta-\beta+1} > \Lambda(A\alpha)^{\beta\lambda}M^{(\alpha-1)\beta\lambda}(1+\theta\frac{1-\alpha}{\alpha}M)^{\lambda}\omega_{t-1}^{(\alpha\beta-\beta+1)\lambda}. \tag{11}$$

Condition (11) means that the marginal value of work is larger than the marginal value of retirement; hence, under this condition, the agent will choose to work full time in the second period.

Case b and case c are similar to the problems in Matsuyama (2006). Now, we consider case d, which is quite different from Matsuyama (2006). Under this case, the optimal capital-labor ratio k_t can be derived similarly as

$$k_t = \frac{K_t}{1+\theta e_t} = \frac{\omega_{t-1} - c_{t-1}^y}{1+\theta e_t}. \tag{12}$$

With the assumption of the Cobb-Douglas production function, and combining equations (3), (12), and (2a), we obtain

$$\begin{aligned}&\Lambda\lambda(A\alpha)^{\beta(\lambda-1)}(1-\beta(1-\alpha))k_t^{\beta(\alpha-1)(\lambda-1)}(k_t+\frac{\alpha\omega_{t-1}}{\theta(1-\alpha)})(\omega_{t-1}-(1-\alpha)k_t)^{\lambda-1}\\&-\Lambda(\lambda+1)(A\alpha)^{\beta(\lambda-1)}\frac{\alpha}{\theta(1-\alpha)}k_t^{\beta(\alpha-1)(\lambda-1)}(\omega_{t-1}-(1-\alpha)k_t)^{\lambda}\\&+(1-\beta(1-\alpha))^{\lambda}k_t=0.\end{aligned} \tag{13}$$

It is difficult to derive the explicit solution of (13)[3]. However, from equation (13), we know that there exists a differentiable function ϕ, such that $k_t=\phi(\omega_{t-1})$. Furthermore, from $\omega_t=(1-\alpha)Ak_t^{\alpha}$, we can derive the dynamic equation for ω_t.

From $R_t=\alpha Ak_t^{\alpha-1}=\alpha A(\phi(\omega_{t-1}))^{\alpha-1}$ and equation (3), we can express the working time choice e_t as a function of ω_{t-1}, $e_t=\tau(\omega_{t-1})$, where τ is a differentiable function.

Therefore, under $1<\lambda<\dfrac{\alpha}{(1-\alpha)\theta\beta}$, we can obtain the transition path for the wage:

$$\omega_t=\Pi(\omega_{t-1})=\begin{cases}(1-\alpha)M^{\alpha}A(\omega_{t-1})^{\alpha}, & \text{if } \omega_{t\text{-}1}\in(0,\omega_c^-]\\(1-\alpha)A(\phi(\omega_{t-1}))^{\alpha}, & \text{if } \omega_{t\text{-}1}\in(\omega_c^-,\omega_c^+),\\(1-\alpha)\beta^{\alpha}A(\omega_{t-1})^{\alpha}, & \text{if } \omega_{t\text{-}1}\in[\omega_c^+,\infty)\end{cases} \tag{14}$$

where ϕ is a differentiable function and

$$\omega_c^-=[\frac{\theta(1-\alpha)(A\alpha)^{\beta(1-\lambda)}M^{1-\beta(1-\alpha)(1-\lambda)}}{(\alpha^{1/\lambda}+\theta(1-\alpha)M\alpha^{(1-\lambda)/\lambda})^{\lambda}\Lambda}]^{\frac{1}{(\lambda-1)(1-\beta(1-\alpha))}},$$

$$\omega_c^+=[\frac{\theta(1-\alpha)(A\alpha)^{\beta(1-\lambda)}\beta^{1-\beta(1-\alpha)(1-\lambda)}}{(\alpha-\lambda(1-\alpha)\beta\theta)\Lambda}]^{\frac{1}{(\lambda-1)(1-\beta(1-\alpha))}}.$$

Note that in Matsuyama (2006), the wage transition path is quite different:

$$\omega_t=\Psi(\omega_{t-1})=\begin{cases}(1-\alpha)(\beta^*)^{\alpha}A(\omega_{t-1})^{\alpha}, & \text{if } \omega_{t\text{-}1}\in(0,\omega_m^-]\\(1-\alpha)(\Omega^{\alpha}A^{\beta-1})^{\frac{\mu-1}{1-\beta(1-\alpha)}}(A(\omega_{t-1}))^{\alpha}, & \text{if } \omega_{t\text{-}1}\in(\omega_m^-,\omega_m^+),\\(1-\alpha)\beta^{\alpha}A(\omega_{t-1})^{\alpha}, & \text{if } \omega_{t\text{-}1}\in[\omega_m^+,\infty)\end{cases} \tag{15}$$

where $\mu=\dfrac{\lambda}{\alpha+(1-\alpha)(1-\beta(1-\lambda))}>1$, $\beta^*=\dfrac{\beta}{1+\theta+(1-\beta)(1-\alpha)\theta/\alpha}<\beta$, $\Omega=(\dfrac{\Lambda\alpha^{1-\beta(1-\lambda)}}{\theta(1-\alpha)})^{\frac{1}{\lambda-1}}$,

and

[3]An explicit solution can be found when $\lambda=1$; however, $\lambda=1$ is not an interesting case for given economic facts.

$$\omega_m^- = \frac{(\beta^*)^{1/(\mu-1)}}{(A^\beta\Omega)^{1/(1-\beta(1-\alpha))}} < \omega_m^+ = \frac{(\beta)^{1/(\mu-1)}}{(A^\beta\Omega)^{1/(1-\beta(1-\alpha))}}.$$

We can easily find that $\omega_c^+ > \omega_m^+$, $\omega_c^- < \omega_m^-$, and analyze the transition path for wage dynamics.

3. Steady State

In this section, we analyze the characteristics of a steady-state economy. First, we consider the existence of a steady state. For comparison, we consider both the 0-1 endogenous retirement choice and the continuous retirement choice.

3.1. 0-1 Endogenous retirement choice

First, we consider Matsuyama's (2006) economy with 0-1 endogenous retirement choices. The steady-state value for the wage rate, ω^*, is reached when $\omega_t = \omega_{t-1}$, and we have the following results for the existence of a steady state.

Proposition 1. *Under the Cobb-Douglas production function and the specified utility function in equation (1), the wage path will be evaluated as equation (15), and a steady state exists. Furthermore, multiple steady states exist if and only if*

$$(\beta^*)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}} \geq [(1-\alpha)A]^{\frac{1}{1-\alpha}}(A^\beta\Omega)^{\frac{1}{1-\beta(1-\alpha)}} \geq (\beta)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}}, \tag{16}$$

where β^*, Ω*, and* β *are defined as before.*

Proof: It is easy to verify that Ψ is continuous on $(0,\infty)$. We define

$$g(\omega_{t-1}) = \Psi(\omega_{t-1}) - \omega_{t-1}.$$

Because $g(\omega)$ is concave in $(0,\omega_m^-]$ and $\lim_{\omega_{t-1}\to 0^+} g'(\omega_{t-1}) = \infty$, there exists $\omega_1 \in (0,\omega_m^-]$ such that $g(\omega_1) > 0$. On the other hand, $g(\omega)$ is concave in $[\omega_m^+,\infty)$ and $\lim_{\omega_{t-1}\to\infty} g'(\omega_{t-1}) = -1$, there exists $\omega_2 \in [\omega_m^+,\infty)$ such that $g(\omega_2) < 0$. Therefore, there must exist $\omega \in (0,\infty)$ such that $g(\omega) = 0$. In other words, a steady state exits.

Next, we prove that condition (16) is a necessary and sufficient condition for the existence of multiple steady states.

First, we prove the sufficiency of condition (16). Because

$$(\beta^*)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}} \geq [(1-\alpha)A]^{\frac{1}{1-\alpha}}(A^\beta\Omega)^{\frac{1}{1-\beta(1-\alpha)}},$$

we have

$$\frac{\Psi(\omega_m^-)}{\omega_m^-} = (1-\alpha)A^{1+\frac{\beta(1-\alpha)}{1-\beta(1-\alpha)}}\Omega^{\frac{\beta(1-\alpha)}{1-\beta(1-\alpha)}}(\beta^*)^{\frac{\alpha\mu-1}{\mu-1}} \le 1,$$

namely $g(\omega_m^-) \le 0$.

On the other hand, because $[(1-\alpha)A]^{\frac{1}{1-\alpha}}(A^{\beta}\Omega)^{\frac{1}{1-\beta(1-\alpha)}} \ge (\beta)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}}$, we have $g(\omega_m^+) \ge 0$.

Therefore, there exits $\omega_1^* \in (0, \omega_m^-]$, such that $g(\omega_1^*) = 0$. Also, there exists $\omega_2^* \in [\omega_m^+, \infty)$, such that $g(\omega_2^*) = 0$. Because $\omega_m^- < \omega_m^+$, we have $\omega_1^* < \omega_2^*$. At least two steady states exist.

Now, we turn to the necessity of condition (16). It is easy to see that Ψ has, at most, one fixed point in either $(0, \omega_m^-]$ or $[\omega_m^+, \infty)$. If multiple steady states exist, then there must be at least two fixed points of Ψ in $(0, \infty)$. We denote them as ω_3^* and ω_4^*, and, without loss of generality, we assume $\omega_3^* < \omega_4^*$. We consider two cases: $\alpha\mu \ne 1$ and $\alpha\mu = 1$.

Case A: $\alpha\mu \ne 1$

In this case, Ψ has, at most, one fixed point in (ω_m^-, ω_m^+), and

$\xi(\omega_{t-1}) = (1-\alpha)(\Omega^{\alpha}A^{\beta-1})^{\frac{\mu-1}{1-\beta(1-\alpha)}}(A(\omega_{t-1}))^{\alpha}$ has, at most, one fixed point in $(0, \infty)$. We consider the following cases.

1) $\omega_3^* \in (0, \omega_m^-]$ and $\omega_4^* \in (\omega_m^-, \omega_m^+)$.

In this case, we know that $g(\omega_m^-) \le 0$. Furthermore, we have $g(\omega_m^-) < 0$, because both ω_m^- and ω_4^* are the fixed points of function $\xi(\omega_{t-1})$ in $(0, \infty)$ when $g(\omega_m^-) = 0$, which leads to a contradiction.

However, because $g(\omega) = 0$ and $g'(\omega) = 0$ cannot have a solution in (ω_m^-, ω_m^+), we have $g'(\omega_4^*) \ne 0$. Furthermore, from $g(\omega_m^-) < 0$, and because $\xi(\omega_{t-1})$ has only one fixed point in $(0, \infty)$, we have $g'(\omega_4^*) > 0$. In turn, we have $g(\omega_m^+) > 0$.

$g'(\omega_4^*) > 0$ and $g(\omega_m^+) > 0$ lead to condition (16).

2) $\omega_3^* \in (0, \omega_m^-]$ and $\omega_4^* \in [\omega_m^+, \infty)$.

In this case, we have $[(1-\alpha)(\beta^*)^{\alpha}A]^{1/(1-\alpha)} \le \omega_m^-$ and $[(1-\alpha)(\beta)^{\alpha}A]^{1/(1-\alpha)} \ge \omega_m^+$, which lead to condition (16).

3) $\omega_3^* \in (\omega_m^-, \omega_m^+)$ and $\omega_4^* \in [\omega_m^+, \infty)$.

This case is similar to 1); we can easily prove that condition (16) is satisfied.

Case B: $\alpha\mu = 1$.

1) If $(1-\alpha)(\Omega^{\alpha} A^{\beta-1})^{\frac{\mu-1}{1-\beta(1-\alpha)}} A^{\mu} = 1$, then $(1-\alpha)(\beta^{*})^{\alpha} A(\omega_m^{-})^{\alpha} = \omega_m^{-}$ and $(1-\alpha)(\beta)^{\alpha} A(\omega_m^{+})^{\alpha} = \omega_m^{+}$. Therefore, we have $\beta = \beta^{*}$, which is a contradiction.

2) If $(1-\alpha)(\Omega^{\alpha} A^{\beta-1})^{\frac{\mu-1}{1-\beta(1-\alpha)}} A^{\mu} \neq 1$, then there is no fixed point for Ψ in $(\omega_m^{-}, \omega_m^{+})$. Therefore, $\omega_3^{*} \in (0, \omega_m^{-})$ and $\omega_4^{*} \in [\omega_m^{+}, \infty)$, and $[(1-\alpha)(\beta^{*})^{\alpha} A]^{1/(1-\alpha)} < \omega_m^{-}$, $[(1-\alpha)(\beta)^{\alpha} A]^{1/(1-\alpha)} > \omega_m^{+}$, and

$$(\beta^{*})^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}} > [(1-\alpha)A]^{\frac{1}{1-\alpha}} (A^{\beta}\Omega)^{\frac{1}{1-\beta(1-\alpha)}} > (\beta)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}}.$$

Therefore, condition (16) is a necessary condition for the existence of multiple steady states.

Q.E.D.

From proposition 1, we know that without assumption (16) a unique steady state exists. Under assumption (16), multiple steady states exist.

Furthermore, we can find that when

$$(\beta^{*})^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}} = [(1-\alpha)A]^{\frac{1}{1-\alpha}} (A^{\beta}\Omega)^{\frac{1}{1-\beta(1-\alpha)}} > (\beta)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}}$$

or

$$(\beta^{*})^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}} > [(1-\alpha)A]^{\frac{1}{1-\alpha}} (A^{\beta}\Omega)^{\frac{1}{1-\beta(1-\alpha)}} = (\beta)^{\frac{1}{\mu-1}-\frac{\alpha}{1-\alpha}},$$

there are two steady states.

If $g(\omega_m^{-}) < 0$ and $g(\omega_m^{+}) > 0$, then three steady states exist, and they are located in $(0, \omega_m^{-})$, $(\omega_m^{-}, \omega_m^{+})$, and (ω_m^{+}, ∞), respectively. Because $\alpha\mu > 1$ always holds in $(\omega_m^{-}, \omega_m^{+})$, there are multiple steady states. The steady state in this range will be unstable, and the other two steady states with the associated retirement choices are stable. Therefore, the retirement choice depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose to retire completely; if the initial capital stock is lower than a critical capital stock, then the individual will choose to work full time.[4]

3.2. Continuous endogenous retirement choice

In this section, we consider the case of continuous retirement choice in the second

[4]The wage path in $(\omega_m^{-}, \omega_m^{+})$ is strictly convex because $\alpha\mu > 1$. This can be found in Matsuyama (2006).

period. Similarly, the steady-state value for the wage rate, ω^*, is reached when $\omega_t = \omega_{t-1}$. To guarantee the existence of a steady state, we assume that

$$1 < \lambda < \frac{\alpha}{(1-\alpha)\theta\beta}. \tag{17}$$

In general, we have the following result.

Proposition 2. *If individuals choose retirement continuously in the second period, then there exists a steady state under condition (17). Furthermore, if*

$$\begin{aligned}&(\frac{1}{\beta}-\lambda\theta\frac{1-\alpha}{\alpha})\beta^{\beta(\alpha-1)(\lambda-1)}(\omega_1^*)^{(\lambda-1)(1-\beta(1-\alpha))} \geq A^{\beta(1-\lambda)}\alpha^{\beta(1-\lambda)-1}\theta(1-\alpha)/\Lambda \\ &\geq (\frac{1}{M}+\theta\frac{1-\alpha}{\alpha})^{\lambda} M^{(\lambda-1)(1-\beta(1-\alpha))}(\omega_2^*)^{(\lambda-1)(1-\beta(1-\alpha))},\end{aligned} \tag{18}$$

where $\omega_1^* = ((1-\alpha)A\beta^{\alpha})^{1/(1-\alpha)}$ *and* $\omega_2^* = ((1-\alpha)AM^{\alpha})^{1/(1-\alpha)}$*, then there exist multiple steady states in the continuous model.*

However, if $\Pi(.)$ *has at most one fixed point in* $[\omega_c^-, \omega_c^+]$ *and* $\Pi'(\omega^*) \neq 1$ *for any* $\omega^* \in (\omega_c^-, \omega_c^+)$ *satisfies* $\Pi(\omega^*) = \omega^*$*, then condition (18) is also a necessary condition for the existence of multiple steady states.*

Proof: Similar to the discussions for proposition 1, we can easily prove that a steady state exists when condition (17) is satisfied.

Note that if $M = \beta^*$, then $\omega_t = \Pi(\omega_{t-1})$, which has the same path as $\omega_t = \Psi(\omega_{t-1})$, both in $(0, \omega_c^-]$ and $[\omega_c^+, \infty)$.

Because

$$(\frac{1}{\beta}-\lambda\theta\frac{1-\alpha}{\alpha})\beta^{\beta(\alpha-1)(\lambda-1)}(\omega_1^*)^{(\lambda-1)(1-\beta(1-\alpha))} \geq A^{\beta(1-\lambda)}\alpha^{\beta(1-\lambda)-1}\theta(1-\alpha)/\Lambda,$$

we have $\omega_1^* \geq \omega_c^+$.

Also, $A^{\beta(1-\lambda)}\alpha^{\beta(1-\lambda)-1}\theta(1-\alpha)/\Lambda \geq (\frac{1}{M}+\theta\frac{1-\alpha}{\alpha})^{\lambda} M^{(\lambda-1)(1-\beta(1-\alpha))}(\omega_2^*)^{(\lambda-1)(1-\beta(1-\alpha))}$, which implies $\omega_2^* \leq \omega_c^-$.

Therefore, at least two steady states, ω_1^* and ω_2^*, exist in $(0, \infty)$.

The proof of the necessity of condition (18) is similar to the proof of proposition 1 if $\omega_t = \Pi(\omega_{t-1})$ has, at most, one fixed point in $[\omega_c^-, \omega_c^+]$ and $\Pi'(\omega^*) \neq 1$ for any $\omega^* \in (\omega_m^-, \omega_m^+)$, which satisfies $\Pi(\omega^*) = \omega^*$.

Q.E.D.

From proposition 2, we know that at least one steady state exists when condition (17) holds.

Remark: It is easy to prove that condition (18) is sufficient to get condition (16) when $\Pi(.)$ has, at most, one fixed point in $[\omega_c^-, \omega_c^+]$ and $\Pi'(\omega^*) \neq 1$ for any $\omega^* \in (\omega_c^-, \omega_c^+)$, which satisfies $\Pi(\omega^*) = \omega^*$. Therefore, with the same parameters, if there are multiple steady states in the continuous endogenous retirement choice model, then multiple steady states must exist in the Matsuyama's (2006) 0-1 endogenous retirement choice model.

Moreover, under some specified parameters, we find that two steady states may exist in (ω_c^-, ω_c^+), one of them an unstable steady state and the other one a stable steady state. Therefore, partial retirement is possible in the second period, and the retirement choice depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level (including the choice of total retirement); if the initial capital stock is lower than a critical capital stock, then the individual will choose a relatively lower retirement level (including the choice of full-time working). This is different from the retirement choice decision in Matsuyama (2006).

4. Numerical solution

To demonstrate the above solutions more clearly, we present a numerical simulation for our model.

Parameter description

There are six parameters in this model: the parameters in utility Λ and λ, the parameters of technology A and α, the efficiency of the physical labor of the old θ, and time preference β. As in de la Croix and Michel (2002), we choose the baseline parameters as follows.

Table 1: Parameter values

Parameter	Λ	λ	A	α	β	θ
Value	1.0	2.0	2.0	$1/3$	$3/13$	0.85

In Table 1, $A = 2.0$ is set to obtain steady state capital around 1000, which could make the difference among the cases more easily visible, and $\alpha = \frac{1}{3}$, $\beta = \frac{3}{13}$ is chosen according to standard real business cycle models.[5] $\theta = 0.85$ implies that one unit of physical labor supplied by the older generation is equivalent to 0.85 units of the physical labor of younger people in the standard setup.

Unique steady state

We can easily obtain the unique steady state by using the baseline parameters. Figures 1 and 2 present the wage paths for Matsuyama's (2006) 0-1 endogenous-retirement choice model and the continuous retirement choice model, respectively.

(Figure 1 and Figure 2 about here)

From Figure 1, we can see that the unique steady-state wage and the choice of working time in the second period are $\omega^* = 0.74$ and $e^* = 0$. We can easily prove that the steady state is stable. The associated retirement choice is that the individual chooses to retire completely. Figure 2 shows the steady state values for wage and the choice of working time in the second period as $\omega^* = 0.69$ and $e^* = 0.070$, and it is stable. Therefore, the individual will choose to work part time.

Although we use the same parameter values, we get different steady states. In the 0-1 retirement choice model, people would retire totally; however, we get an interior steady state in which people choose to work part time.

Multiple steady states

Letting A=10, α=0.74, λ=14.25, and the other parameters remain the same as in Table 1, Figure 3 shows the wage path under the 0-1 retirement choice model. From Figure 3, we can see that three steady states exist: $\omega_1^* = 0.076$, $\omega_2^* = 0.44$, and $\omega_3^* = 0.61$. The associated working time choices in the second period are $e_1^* = 1$, $e_2^* = 0.11$, and $e_3^* = 0$, respectively. They are located in $(0, \omega_m^-]$, (ω_m^-, ω_m^+), or $[\omega_m^+, \infty)$, respectively.

(figure 3 about here)

We can easily prove that the first and third steady states are stable, and the second steady state is unstable. Therefore, the economy will converge to the steady state with a higher wage level when the initial wage level is larger than $\omega_2^* = 0.44$ (with a relatively higher capital stock), and the individual will choose to retire completely in the second

[5]In a standard RBC model, the quarterly psychological discount factor is 0.99; thus the 30-year discount factor $\beta' = 0.99^{120} = 0.30$. Here, an individual's lifetime utility has the form $U(c_{t-1}, c_t) = \log c_{t-1} + \beta' \log c_t$. After some simple computations, we get $\beta = \frac{3}{13}$ in our framework.

period. However, the economy will converge to the steady state with a lower wage level when the initial wage level is lower than $\omega_2^* = 0.44$ (with a relatively lower capital stock), and the individual will choose to work full time in the second period.

(Figure 4 about here)

To derive multiple steady states, we let A=10, $\alpha = 0.74$, and $\lambda = 14.25$, and the other parameters remain as listed in Table 1. Figure 4 presents the wage path under the continuous retirement choice. We know that there are three steady states: $\omega_1^* = 0.076$, $\omega_2^* = 0.53$, and $\omega_3^* = 0.60$. The associated working time choices in the second period are $e_1^* = 1$, $e_2^* = 0.043$, and $e_3^* = 0.0046$, respectively. The first steady state is located in $(0, \omega_c^-]$, and the other two are located in (ω_c^-, ω_c^+).

Also, we can easily prove that the first and third steady states are stable, and the second steady state is unstable. Therefore, the economy will converge to the steady state with a higher wage level when the initial wage level is larger than $\omega_2^* = 0.53$ (with relatively higher capital stock), and the individual will choose retire partially in the second period. However, the economy will converge to the steady state with a lower wage level when the initial wage level is lower than $\omega_2^* = 0.53$ (with relatively lower capital stock), and the individual will choose to work full time in the second period.

Because $\alpha\mu > 1$ always holds in (ω_m^-, ω_m^+) under 0-1 retirement choice when there are multiple steady states. The steady state in this range will be unstable, and the other two steady states with the associated retirement choices $e_1^* = 1$ and $e_3^* = 0$ are stable. Therefore, the steady state capital stock determines an individual's retirement choice: whether to retire totally or work full time. This can be found in Matsuyama (2006).

However, the wage path in (ω_c^-, ω_c^+) will not always be convex when the individual chooses retirement continuously in the second period. If two steady states exist in (ω_c^-, ω_c^+), then partial retirement will be a stable steady state. The individual's retirement choice depends on the initial capital stock (wage level). If the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level; if the initial capital stock is lower than a critical capital stock, then the individual will choose a relatively lower retirement level. We also notice that our analysis of multiple equilibria and steady states has presented interesting results closely related to the earlier studied by Kiminori Matsuyama (1992).

5. Conclusions

We extend Matsuyama's (2006) 0-1 retirement choice model to a framework with continuous retirement choice to examine an individual's optimal saving-retirement choices in an aggregate economy. In this new setup, we use the same utility function given by Matsuyama (2006), and we obtain the optimal saving-retirement decisions explicitly. However, we could only obtain the implicit forms of the wage path and the optimal retirement decision when the model had interior solutions.

We find that with the same parameter values as Matsuyama's (2006), there is a larger domain for the wage path to have an interior solution, on which $e_t \in (0,1)$, in the continuous model than there is in Matsuyama's model.

Furthermore, we find two conditions to guarantee the existence of multiple steady states for wage paths in both Matsuyama's model and in our continuous endogenous retirement model. Under some conditions, we find that it is easier to generate multiple steady states in the discrete model than in the continuous retirement choice model. The steady state with $e \in (0,1)$ is not stable in Matsuyama's (2006) model; therefore, from any initial state, the economy could only converge to a steady state in which people have to work full time or retire completely when they are old, i.e., part-time work is not allowed in the steady state of the 0-1 retirement choice model when the wage path has multiple steady states. However, with the consideration of continuous retirement choice in this paper, the steady state $e \in (0,1)$ may be a stable steady state, and partial retirement is allowed. The retirement decision depends on the initial capital stock: if the initial capital stock is larger than a critical capital stock, then the individual will choose a relatively higher retirement level and the economy will converge to an economy with relatively higher capital accumulation; if the initial capital stock is lower than a critical capital stock, the individual will choose a relatively lower retirement level and the economy will converge to an economy with relatively lower capital accumulation.

We also present a numerical simulation for our analysis. In Figures 3 and 4, we derive three steady states under Matsuyama's (2006) 0-1-retirement choice and the continuous retirement choice, respectively. However, under the continuous retirement choice, we obtain two steady states in (ω_c^-, ω_c^+) and show that one of them is stable. Therefore, part-time work is possible in the continuous retirement choice model, whereas it is impossible in Matsuyama's (2006) model.

However, to generate multiple steady states, this paper requires the parameter λ to be a large value in both two models. Our future work will concentrate on generating multiple steady states with empirically plausible parameter values by introducing other discounting frameworks such as hyperbolic discounting. Also, we may include many other macroeconomic topics in this framework. For instance, we could extend the paper with a consideration of social security to study the effects of social security on the retirement choice.

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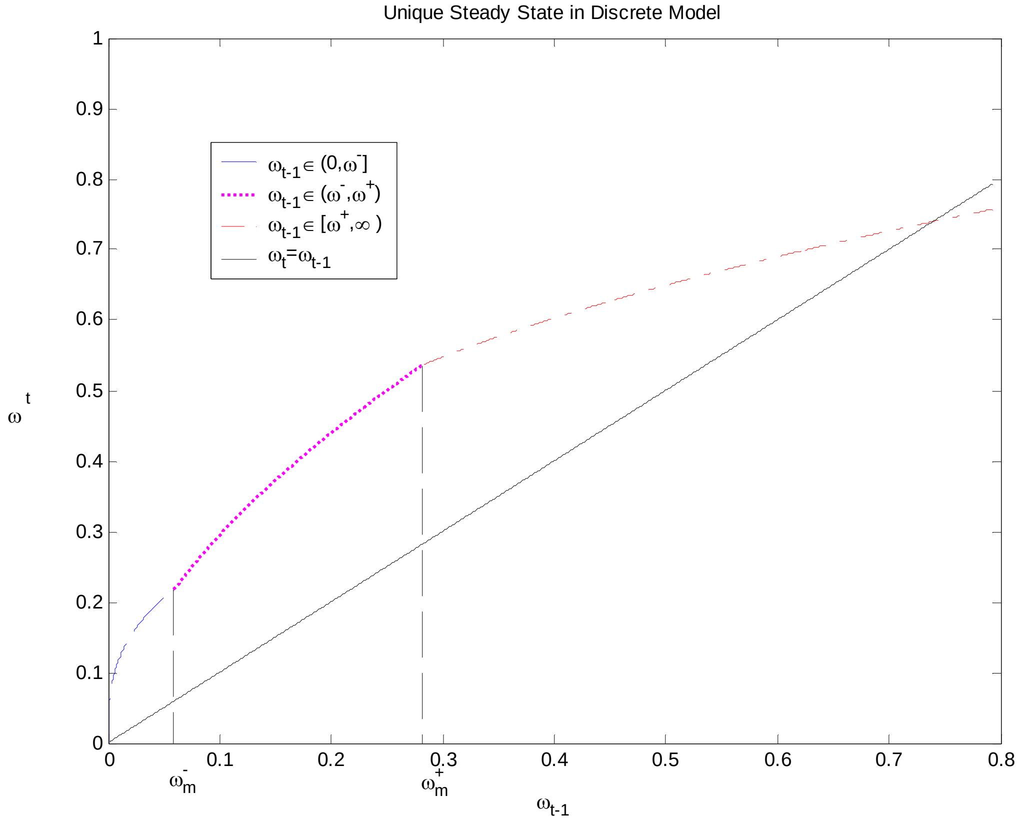

Figure 1: Unique steady state in the 0-1 retirement choice model.

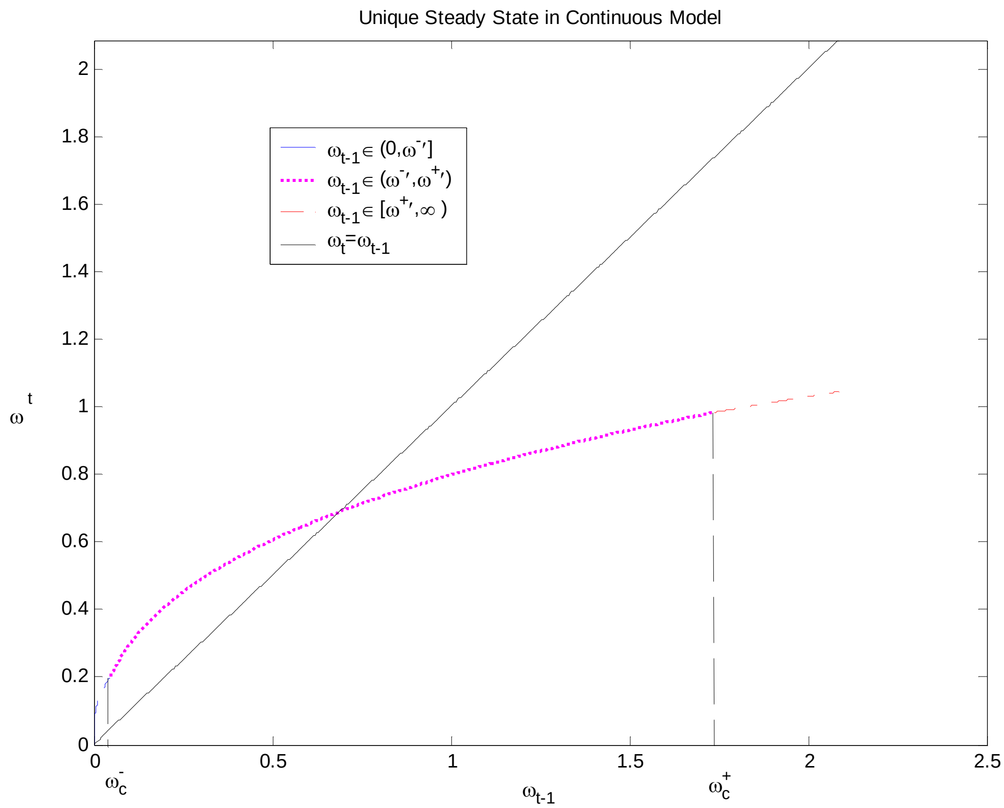

Figure 2: Unique steady state in the continuous retirement choice model.

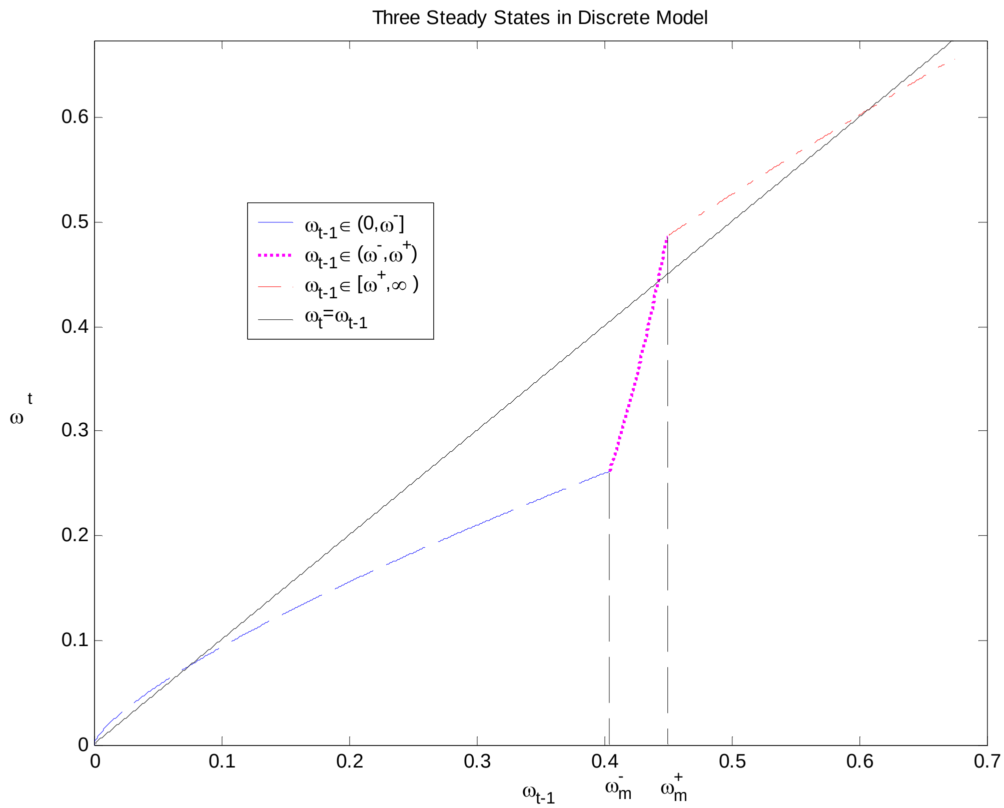

Figure 3: Multiple steady states in the 0-1 retirement choice model.

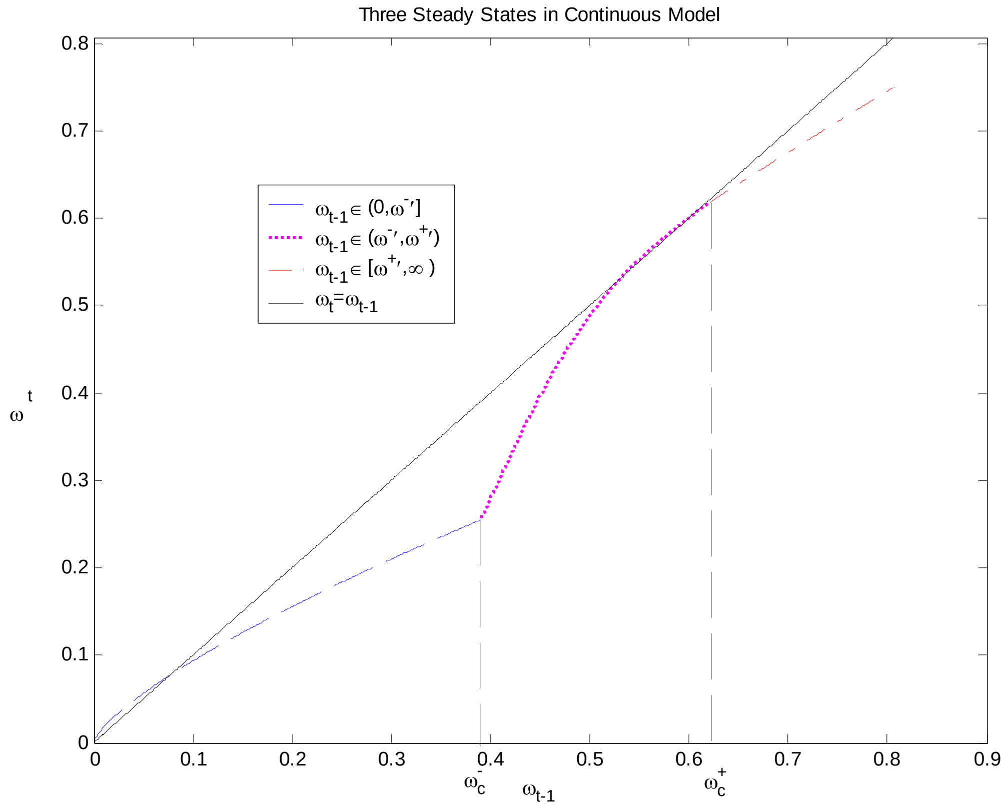

Figure 4: Multiple steady states in the continuous retirement choice model.