Competitive Equilibrium in an Overlapping Generations Model with Production Loans*

Dihai Wang†
Fudan University

Gaowang Wang‡
Central University of Finance and Economics

Heng-fu Zou§
Central University of Finance and Economics

June, 2012

Abstract

The paper shows that there do exist two kinds of steady state equilibria in the overlapping generations models with consumption and production loans, similar to the pure exchagne economies examined by Gale (1973). Furthermore, the local stability properties of these two (kinds of) steady states are also investigated.

Keywords: Multiple steady state paths, Overlapping generations, Production loans

JEL Classification Numbers: E21, O41

---

*Wang Dihai thanks the support of Shanghai Leading Academic Discipline Project (Project Number: B101). Wang Gaowang thanks the support of China Postdoctoral Science Foundation (Project Number: 2012M510734).

†School of Economics, Fudan University, Shanghai, China. Email: wangdihai@fudan.edu.cn.

‡China Economics and Maganagement Acedamy, Central University of Finance and Economics, Beijing, China. Email: wanggaowang@gmail.com

§China Economics and Maganagement Acedamy, Central University of Finance and Economics, Beijing, China. Email: zouhengfu@gmail.com

1 Introduction

In his seminal paper, Samuelson (1958) constructs the overlapping generation model with consumption loans, examines the determination of interest rates and suggests for further research the OLG model with capital.[1] By introducing production employing a durable capital good into the Samuelson (1958) model, Diamond (1965) examines the long-run competitive equilibrium in a growth model with both consumption and production loans and explores the effects on this equilibrium of government debt.

Gale (1973) find two kinds of steady state paths and examines their global stability properties in a Samuelson-type pure exchagne economy. And whether there exist multiple kinds of steady state paths in the Diamond (1965) model with production loans is an open question. The paper wants to answer the question. By utilizing an overlapping generations model with both consumption and production loans, the paper shows that there do exist two kinds of steady state paths. Furthermore, the local stability of those multiple (kinds of) steady states is examined.

The paper is organized as follows. Section 2 describes the model. Section 3 examines competitive quilibrium and multiple steady state paths. By executing the phase diagram analysis, section 4 investigates the local stability of the steady states. The conclusion is in section 5. The proof appears in the Appendix.

2 Model

The economy being considered is assumed to have an infinite future. Each individual lives for two periods, working in the first period (young) while being retired in the second (old). The birth rate of the population is a given constant, n. And the total number of people in period t is the sum of the young people $L_t = L_0(1+n)^t$ and the old people $L_{t-1} = L_0(1+n)^{t-1}$, namely,

$$P_t \equiv L_t + L_{t-1} = L_0(2+n)(1+n)^{t-1},$$

which tells that the population growth rate is also n. It is assumed that all consumers are identical as to preferences and income streams and differ only

[1] It is well known that Samuelson's model supplies an analytical framework for the existence of money.

as regards their dates of birth. A consumer who is born in period t works with wage w_t, consumes $c_{y,t}$ and saves $s_t(= w_t - c_{y,t})$ in period t; and in period $t+1$ he consumes the gross interest income of the savings of period t, i.e., $c_{o,t+1} = (1 + r_{t+1})s_t$ and dies. His utility function is

$$U(c_{y,t}, c_{o,t+1}) = u(c_{y,t}) + \beta u(c_{o,t+1}), \tag{1}$$

where β is the time preference rate, and $u(c)$ is a strictly positive, strictly increasing, and strictly concave function satisfying the Inada conditions. Then, given the prices w_t and r_{t+1}, the maximization problem of the consumer born in period t is to maximize (1) subjecting to the budget constraints

$$c_{y,t} + s_t = w_t, c_{o,t+1} = (1 + r_{t+1})s_t. \tag{2}$$

The optimality condition is

$$u'(w_t - s_t) = \beta(1 + r_{t+1})u'((1 + r_{t+1})s_t), \tag{3}$$

which determines implicity the savings function of the consumer who was born in period t as a function of those prices by the strict concavity of u,

$$s_t = s(w_t, r_{t+1}). \tag{4}$$

And then he consumes $c_{y,t} = w_t - s(w_t, r_{t+1})$ in period t and consumes $c_{o,t+1} = (1 + r_{t+1})s(w_t, r_{t+1})$ in period $t+1$.

With the neoclassical production technology $Y_t = F(K_t, L_t)$ and the depreciate rate δ, a representative firm produces the unique commordity in the economy by employing both capital K_t with the rental rate r_t and labor L_t with the wage cost w_t. The total productive capital K_t includes private capital $K_{s,t}$ and public capital $K_{g,t}$, i.e., $K_t = K_{s,t} + K_{g,t}$. It is assumed that public capital has an exogenously given accumulation process with the same rate of return as private capital,

$$K_{g,t+1} = (1 + r_t)K_{g,t}. \tag{5}$$

The labor force in period t must be the young people in period t, L_t. The profit maximization problem of the representative firm is as follows,

$$\Pi_t = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t.$$

With the definitions of the per-labor capital $k_t \equiv \frac{K_t}{L_t}$ and the per-labor production $f(k_t) \equiv F(K_t, L_t)$, the optimality leads to

$$r_t = f'(k_t) - \delta, w_t = f(k_t) - k_t f'(k_t). \tag{6}$$

3 Competitive Equilibrium and Steady States

In order to examine the competitive equilibrium, we should attach the market clearing condition of the commordity market:

$$c_{o,t}L_{t-1} + c_{y,t}L_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t), \tag{7}$$

which tells that in period t the sum of the total consumption $c_{o,t}L_{t-1} + c_{y,t}L_t$ and total investment $K_{t+1} - (1-\delta)K_t$ is equal to the total production $F(K_t, L_t)$. In Samuelson (1958) and Gale (1973)'s pure exchange economies, only consumption loans are considered. And the market-clearing condition is that the total savings in the economy must cancel out to zero in every period. However, the model considered here is a Diamond-type OLG model with both consumption loans and production loans. In every period, the total saving $F(K_t, L_t)-$ $(c_{o,t}L_{t-1} + c_{y,t}L_t)$ is equal to the total investment $(K_{t+1} - (1-\delta)K_t)$ in each period. And the amount of the total saving or investment is exactly the amout of production loans.[2] By substituting the consumption functions, the savings function (4), and the pricing functions (5), we obtain the dynamic accumulation equation of capital per labor:

$$[f'(k_t) + (1-\delta)] \left[s\left(w(k_{t-1}), r(k_t)\right) - (1+n)k_t \right] = (1+n) \left[s\left(w(k_t), r(k_{t+1})\right) - (1+n)k_{t+1} \right]. \tag{8}$$

In order to obtain the steady state equilibria, setting $k_{t+1} = k_t = k^*$, $s_{t+1} = s_t = s^*$ in equation (8) gives rise to

$$[f'(k^*) - (\delta + n)] \left[s^* - (1+n)k^* \right] = 0, \tag{9}$$

from which we can derive the following two kinds of steady states

$$f'(k_1^*) - \delta = n, \;\; s\left(w(k_2^*), r(k_2^*)\right) = (1+n)k_2^*. \tag{10}$$

[2] Certainly, if the production function is just a linear funtion about labor and the total investment is zero in each period, the model here will degenerate to the Samuelson (1958) and Gale (1973) models without production loans..

Before presenting the proposition, we examine the two kinds of steady states at first. The steady state capital stock of the first type satisfies the modified golden rule.[3] Furthermore, equations (6) and (9) tell us that $r_1^* \, (= f'(k_1^*) - \delta) = n$, which shows that the equilibrium interest rate depends only on the structure of the population (i.e., the population growth rate) and has nothing to do with the technology and preference of the economy. The equilibrium interest rate determined by the structure of the population is called "the biological rate of interest" defined by Samuelson (1958) and followed by Gale (1973). To comprehend the steady state of the second hand, we multiply both sides of equation (9) by L_{t-1},

$$s\left(w(k_2^*), r(k_2^*)\right) L_{t-1} = k_2^* L_t (= K^*). \tag{11}$$

The left-hand side of equation (11) is the total private savings in period t, and the right-hand side stands for the aggregate capital in period t. Their equality tells that the public capital (or debt) is zero in the economy. And this second kind of steady states are called "balanced" ones in the sence of zero public capital.[4]Obviously, there must be an equilibrium interest rate $r_2^* \, (= f'(k_2^*) - \delta)$ associated with the balanced steady states. Similar to Gale's (1973) pure exchange economy, we have the following proposition and definition.

Proposition 1 *In the overlapping generations model with production loans, there exist two kinds of steady states: one is the golden-rule steady state with "the biological rate of interest", and the other is the balanced steady states without any role of public capital (or debt).*[5]

Definition 2 *A model will be called classical (Samuelson) if $k_1^* > k_2^*$ ($k_1^* < k_2^*$).*

The definition is a generalization (to the economy with production) of Gale's (1973) definition which depends on the exogenous parameters in pure

---

[3]This is the standard result on the nature of the golden rule path, see, e.g., Phelps (1961) and Diamond (1965).

[4]Gale (1973) calls the no-trade (neither borrow nor lend, i.e., aggregate assets are zero) stationary equilibrium "balanced" in pure exchange economies. And this balance is in the sense of balanced accounts, not balanced growth. In this paper, the balanced steady states are those without public assets or debts.

[5]We exclude the case that the two kinds of equilibria coincide (i.e., $r_2^* = n$).

exchange economies. And they are essentially the same. The intuition will be revealed in the next section: in the classical case people consume more and save less in both kinds of stationary equilibria, and in the Samuelson case people consume less and save more in these equilibria.

Theorem 3 *A model is classical (Samuelson) if and only if $r_2^* > n$ ($r_2^* < n$).*
Proof. *It can be easily proved by using (6) and (10). And we omit it.* ■

4 Local Stability of the Equilibria and Phase Diagram Analysis

In this section, we will examine the local stability of these two (kinds) of steady states and their local dynamics. For the second kind of steady state, there may exist multiple equilibrium with different assumptions on the production and utility functions, to which we will not attach importance.[6] Hence, for simplicity, we assume log utility, Cob-Douglas technology, and complete depreciation: $u(c) = \log c$, $Y = AK^{\alpha}L^{1-\alpha}$, and $\delta = 1$.

Substituting the assumptions into the optimality conditions in the above section leads to the savings function

$$s_t = \frac{\beta}{1-\beta} w_t, \tag{12}$$

the consumption functions

$$c_{y,t} = \frac{w_t}{1-\beta}, \; c_{o,t+1} = \frac{\beta(1+r_{t+1})}{1-\beta} w_t, \tag{13}$$

the pricing functions

$$r_t = A\alpha k_t^{\alpha-1} - 1, \; w_t = A(1-\alpha)k_t^{\alpha}, \tag{14}$$

and the two (kinds of) steady states

$$k_1^* = \left(\frac{A\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}, \; k_2^* = \left(\frac{A(1-\alpha)\beta}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}}, \tag{15}$$

[6]In his seminal paper, Samuelson (1958) has pointed out the existence of multiple equilibrium. And Bliss (2008) and Hiraguchi (2012) gives some particular examples of multiple equilibrium in the Diamond capital model numerically or analytically.

where k_1^* is the golden rule level of the steady state and k_2^* is the balanced steady state.

Proposition 4 *A model is classical* (*Samuelson*) *if and only if* $\frac{\alpha}{1-\alpha} > \frac{\beta}{1+\beta}$ $\left(\frac{\alpha}{1-\alpha} < \frac{\beta}{1+\beta}\right)$.

Proof. *The conclusion can be drawn from the relations* $k_1^* > (<)\, k_2^* \Longleftrightarrow r_2^* > (<)\, n \Longleftrightarrow \frac{\alpha}{1-\alpha} > (<)\, \frac{\beta}{1+\beta}$*, which is derived from definition 2, theorem 3, and (15).* ∎

In order to examine the stability of the steady state and the dynamics of the system, we are going to execute the phase diagram analysis. Define $k_{g,t} \equiv \frac{K_{g,t}}{L_t}$ and $k_{s,t} \equiv \frac{K_{s,t}}{L_t}$ as the public capital per labor and private capital per labor. Hence,

$$K_t = K_{s,t} + K_{g,t} \Longleftrightarrow k_t = k_{s,t} + k_{g,t}. \tag{16}$$

On one hand, equations (5), (14) and (16) give rise to

$$k_{g,t+1} = \frac{A\alpha\left(k_{g,t} + k_{s,t}\right)^{\alpha-1}}{1+n} k_{g,t}; \tag{17}$$

On the other hand, by the assumption of complete depreciation, we know that the private capital stock of any time t is the total savings of the old people at time t, namely, $K_{s,t+1} = L_t s_t$. Putting (12), (14), and (16) into the above equation and dividing the derived equation by L_{t+1} turn out to

$$k_{s,t+1} = \frac{A(1-\alpha)\beta\left(k_{g,t} + k_{s,t}\right)^{\alpha}}{(1+n)(1+\beta)}. \tag{18}$$

Equations (17) and (18) can determine the whole dynamics of the model. With the definition of the steady state (k_s^*, k_g^*) which subjects to $k_s^* = k_{s,t+1} = k_{s,t}$ and $k_g^* = k_{g,t+1} = k_{g,t}$, it is straightforward to derive the two (kinds of) steady states from equations (17) and (18):

$$(k_{s1}^*, k_{g1}^*) = \left(\frac{(1-\alpha)\,\beta}{1+\beta}\left(\frac{A}{1+n}\right)^{\frac{1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}, \left(\frac{A\alpha}{1+n}\right)^{\frac{1}{1-\alpha}} - k_{s1}^*\right), \tag{19}$$

$$(k_{s2}^*, k_{g2}^*) = \left(\left(\frac{A\,(1-\alpha)\,\beta}{(1+n)\,(1+\beta)}\right)^{\frac{1}{1-\alpha}}, 0\right), \tag{20}$$

with the associated steady state levels of interest rates $r_1^* = n$, $r_2^* = \frac{(1+n)\alpha((1+\beta))}{(1-\alpha)\beta}$. And it is easy to check that $k_1^* = k_{s1}^* + k_{g1}^*$, and $k_2^* = k_{s2}^* + k_{g2}^*$.

The dynamics of the system composed by equations (17) and (18) can be described in the phase space (k_s, k_g). From equation (17), we know that public capital keeps constant on the straight line AB $\left(\text{i.e., } k_g + k_s = \left(\frac{A\alpha}{1+n}\right)^{\frac{1}{1-\alpha}}\right)$ and the k_s-axis (i.e., $k_g \equiv 0$). Furthermore, if $k_g > 0$ (i.e., public capital), then public capital decreases above the line AB while increases below it; if $k_g < 0$ (i.e., public debt), then public debt increases above the line AB while decreases below it. Meanwhile, from (18), we know that private capital keeps constant on the smooth curve OCD $\left(\text{i.e., } k_g = \left(\frac{(1+n)(1+\beta)}{A(1-\alpha)\beta}\right)^{\frac{1}{\alpha}} k_s^{\frac{1}{\alpha}} - k_s\right)$, which is strictly concave and passes through the origin. It is not hard to know that private capital increases above the curve and decreases below it. If putting these two curves into the same phase space, it is easy to find that the line AB and the curve OCD intersect only once in the space, but with two possibilities: the classical and Samuelson case.

(1) The classical case $\left(\frac{\alpha}{1-\alpha} > \frac{\beta}{1+\beta}\right)$.

In classical case (figure 1), (k_{s1}^*, k_{g1}^*) (E_1 in figure 1) is the golden-rule steady state with "the biological interest rate" $r_1^* (= n)$ and (k_{s2}^*, k_{g2}^*) (E_2 in figure 1) is the balanced steady state with the balanced interest rate $r_2^* (> n)$. And they satisfy $k_{s1}^* > k_{s2}^*$, $k_{g1}^* > k_{g2}^* (= 0)$, and hence $k_1^* > k_{g2}^*$. The direction field of the system in figure 1 shows that the golden rule steady state is locally stable and the balanced steady state is locally saddle-point stable. The rigorous proof of the stability results can be found in the mathematical appendix.

(Insert figure 1 here)

(2) The Samuelson case $\left(\frac{\alpha}{1-\alpha} < \frac{\beta}{1+\beta}\right)$.

Similarly, in Samuelson case (figure 2), (k_{s1}^*, k_{g1}^*) (E_1 in figure 2) is the golden-rule steady state with "the biological interest rate" $r_1^* (= n)$ and (k_{s2}^*, k_{g2}^*) (E_2 in figure 2) is the balanced steady state with the balanced interest rate $r_2^* (< n)$. And they satisfy $k_{s1}^* < k_{s2}^*$, $k_{g1}^* < k_{g2}^* (= 0)$, and hence $k_1^* < k_{g2}^*$. The direction field of the system in figure 2 shows that the golden rule steady state is locally saddle-point stable and the balanced steady state is locally stable. The proof can be also found in the mathematical appendix.

(Insert figure 2 here)

Proposition 5 *In the overlapping generations model with production loans,* *in the classical case the golden-rule steady state is locally stable and the balanced steady state is locally saddle-point stable, however, in Samuelson case the golden-rule steady state is locally saddle-point stable and the balanced steady state is locally stable.*

5 Conclusion

By utilizing an overlapping generations model with both consumption and production loans, the paper shows that there do exist two kinds of steady state paths, similar to the pure exchange economies examined by Gale (1973). Furthermore, the local stability of those multiple (kinds of) steady state is investigated.

6 Appendix

In this appendix, we examine the local stability of the two (kinds of) steady states. Define $\widetilde{k}_{i,t} = k_{i,t} - k_i^*$, $i = g, s$. The dynamic system given by (17) and (18) can be linearized around the steady state as

$$\begin{aligned} \widetilde{k}_{g,t+1} &\simeq k_{gg}\widetilde{k}_{g,t} + k_{gs}\widetilde{k}_{g,t}, \\ \widetilde{k}_{s,t+1} &\simeq k_{sg}\widetilde{k}_{g,t} + k_{ss}\widetilde{k}_{s,t}, \end{aligned}$$

where $k_{gg} = \frac{A\alpha\left(k_g^*+k_s^*\right)^{\alpha-1}}{(1+n)}\left(\frac{(\alpha-1)k_g^*}{\left(k_g^*+k_s^*\right)} + 1\right)$, $k_{gs} = \left(\frac{A\alpha(\alpha-1)\left(k_g^*+k_s^*\right)^{\alpha-2}k_g^*}{(1+n)}\right)$,
$k_{sg} = \frac{A\alpha(1-\alpha)\beta\left(k_g^*+k_s^*\right)^{\alpha-1}}{(1+n)(1+\beta)}$, $k_{ss} = \frac{A\alpha(1-\alpha)\beta\left(k_g^*+k_s^*\right)^{\alpha-1}}{(1+n)(1+\beta)}$.

Since the procedure is similar, we consider only the classical case. In the golden-rule steady state E_1, the matrix form of the linearized system becomes:

$$\begin{bmatrix} \widetilde{k}_{g,t+1} \\ \widetilde{k}_{s,t+1} \end{bmatrix} \simeq \begin{bmatrix} 1-(\alpha-1)\left(1-\frac{(1-\alpha)\beta}{\alpha(1+\beta)}\right) & (\alpha-1)\left(1-\frac{(1-\alpha)\beta}{\alpha(1+\beta)}\right) \\ \frac{(1-\alpha)\beta}{1+\beta} & \frac{(1-\alpha)\beta}{1+\beta} \end{bmatrix} \begin{bmatrix} \widetilde{k}_{g,t} \\ \widetilde{k}_{s,t} \end{bmatrix}$$

Define the Jacobian matrix of the linearized system as M_1 and its two eigenvalues as λ_1, λ_2. Hence, we have

$$\lambda_1 + \lambda_2 = trace\left(M_1\right) = \alpha + \frac{\left(1-\alpha\right)\beta}{\alpha\left(1+\beta\right)},\ \lambda_1\lambda_2 = \det\left(M_1\right) = \frac{\left(1-\alpha\right)\beta}{\alpha\left(1+\beta\right)}.$$

Their solutions are: $\lambda_1 = \frac{(1-\alpha)\beta}{\alpha(1+\beta)} \in (0,1)$, $\lambda_2 = \alpha \in (0,1)$, which show that the golden-rule steady state is locally stable. Similarly, in the balanced steady state E_2, the matrix form of the linearized system is

$$\left[\begin{array}{c} \widetilde{k}_{g,t+1} \\ \widetilde{k}_{s,t+1} \end{array}\right] \simeq \left[\begin{array}{cc} \frac{\alpha(1+\beta)}{(1-\alpha)\beta} & 0 \\ \alpha & \alpha \end{array}\right] \left[\begin{array}{c} \widetilde{k}_{g,t} \\ \widetilde{k}_{s,t} \end{array}\right].$$

Define the Jacobian matrix of the linearized system as M_2 and its two eigenvalues as μ_1, μ_2. Hence,

$$\mu_1 + \mu_2 = trace\left(M_1\right) = \alpha + \frac{\alpha\left(1+\beta\right)}{\left(1-\alpha\right)\beta},\ \mu_1\mu_2 = \det\left(M_1\right) = \frac{\alpha^2\left(1+\beta\right)}{\left(1-\alpha\right)\beta}.$$

The corresponding solutions are: $\mu_1 = \frac{\alpha(1+\beta)}{(1-\alpha)\beta} \in (1,\infty)$, $\mu_2 = \alpha \in (0,1)$, which tell that the balanced steady state is locally saddly-point stable. $\square$

References

[1] Bliss, C., 2008. Multiple Equilibrium in the Diamond Capital Model. Economics Letters 100, 143-145.

[2] Diamond, P., 1965. National Debt in a Neoclassical Growth Model. The Amerian Economic Review 55 (5), 1126-1150.

[3] Gale, D., 1973. Pure Exchange Equilibrium of Dynamic Economic Model. Journal of Economic Theory 6, 12-36.

[4] Hiraguchi, R., 2012. Multiple Equilibrium in the Overlapping Generations Model Revisited. Economics Letters 115, 70-72.

[5] Phelps, E., 1961. The Golden Rule of Accumulation: A Fable for Growthman. The American Economic Review 51, 638-643.

[6] Samuelson, P., 1958. An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. Journal of Political Economy 66, 467-482.

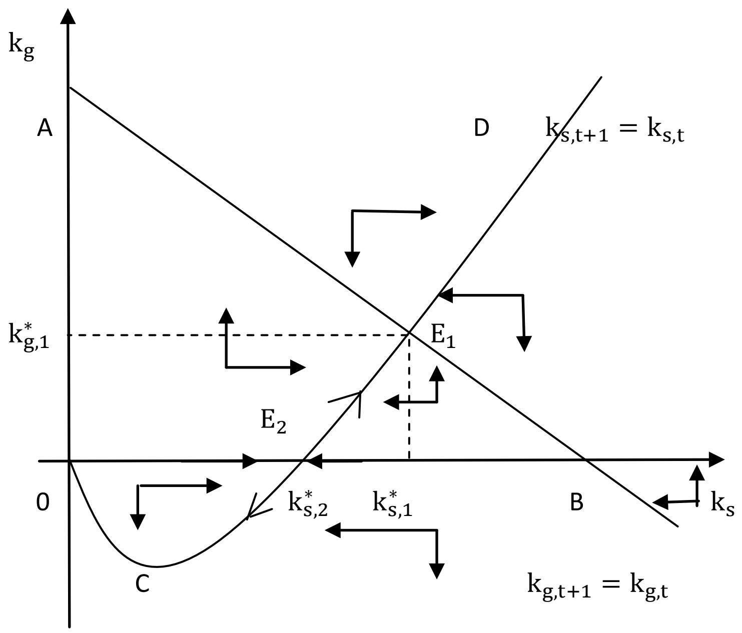

Figure 1: The classical case

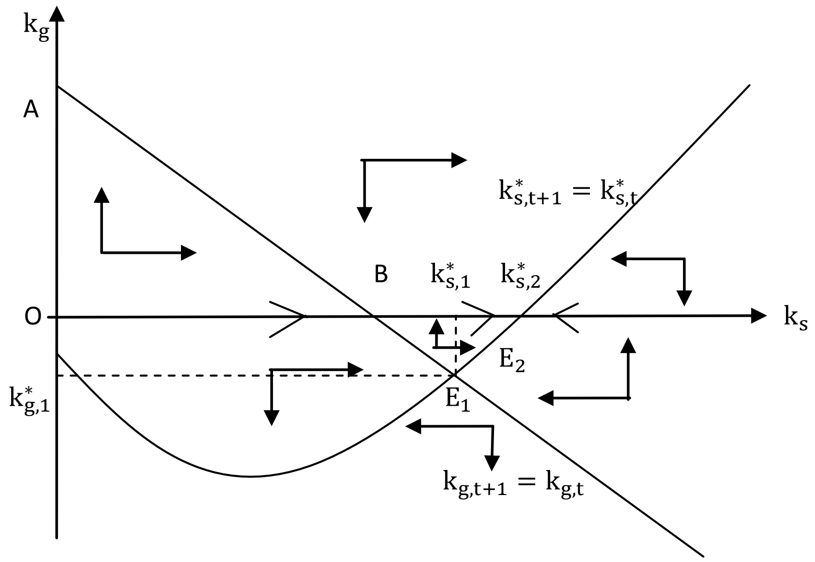

Figure 2: The Samuelson case