The $Z-$Transform Method for Multidimensional Dynamic Economic Systems

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Abstract

This paper uses the Z-transform to develop a method for solving the linearized multidimensional discrete-time systems, which can be used to discuss the effects of policies on economy (including the welfare gains and initial effects on economy) raised by multi-sector perfect-foresight-discrete-time models. Our method is not restricted on the dimension of the dynamic system, and it cannot only analyze the effect of permanent policy change on the economy, but also it can be used to analyze the effect of temporal policy change on the economy. As an application example, we analyze the effects of fiscal policy on the initial economy and social welfare in the discrete-time Uzawa-Lucas model.

Key Words: $Z-$transform; Steady-state comparisons; Multi-sector model.

JEL Classifications: C63, E1, H2.

1 Introduction

Since the 1980s, many methods for steady-state comparisons in perfect-foresight models have been proposed to analyze the welfare effects of fiscal and monetary policies. For example, Judd (1981), Turnovsky (1993), Chamley (1981), and Cui and Gong (2006), among others, have developed different techniques to examine the welfare impact of an exogenous policy perturbation on the steady-state endogenous variables in continuous-time optimization problems. However, as far as we know, few methods are developed to deal with discrete-time optimization problems. Recently, Meijdam and Verhoeven (1998) have used the Z-transform method to examine the comparative dynamics in a discrete-time model, which generates an analytically tractable approximation to the solutions of a two-dimension difference-equation system. However, as in Judd's Laplace-transform approach, the Meijdam-Verhoeven method cannot be applied directly to the multidimensional discrete-time dynamic systems with more than three state and control variables.

This paper is to use the Z-transform to develop a method for solving the linearized multidimensional systems with n (n > 2) state and control variables, which can be used to analyze high-dimension problems in multi-sector dynamic models. The new method in presented in Section 2, and, as an illustration of wide applications of the new method, the effects of capital income taxation on the economy in the discrete-time Uzawa-Lucas model is examined in Section 3. A few remarks are made in Section 4.

2 The Method for Solving Multidimensional System

Consider the following discrete-time, dynamic economic model

$$x_{t+1}^{i} = f_i(x_t^1,\cdots,x_t^{n_1},y_t^1,\cdots,y_t^{n-n_1},\epsilon h_t^1,\cdots,\epsilon h_t^K),\ i=1,\cdots,n_1, \tag{1a}$$

$$y_{t+1}^{j} = g_j(x_t^1,\cdots,x_t^{n_1},y_t^1,\cdots,y_t^{n-n_1},\epsilon h_t^1,\cdots,\epsilon h_t^K),\ j=1,\cdots,n-n_1, \tag{1b}$$

where $f_i : R^{n_1}\times R^{n-n_1}\times R^K \to R$ and $g_j : R^{n_1}\times R^{n-n_1}\times R^K \to R$ are continuous differentiable functions. $x_t^1,\cdots,x_t^{n_1}$ are predetermined variables (or backward-looking variables) with the given initial values $x_0^1, \cdots, x_0^{n_1}$; $y_t^1, \cdots, y_t^{n-n_1}$ are non-predetermined variables (also known as jumping or forward-looking variables); ϵ is a scale parameter, initially equals zero; and for any $l = 1, \cdots, K$, h_t^l is bounded and eventually a constant function of time. For convenience, we denote $x_t = (x_t^1, \cdots, x_t^{n_1})^T$, $y_t = (y_t^1, \cdots, y_t^{n-n_1})^T$, and $h_t = (h_t^1, \cdots, h_t^K)^T$.

Suppose we are initially in a steady state of the system, i.e. initial values $x_0^1, \cdots, x_0^{n_1}$ and $y_0^1, \cdots, y_0^{n-n_1}$ are selected such that

$$f_i(x_0^1, \cdots, x_0^{n_1}, y_0^1, \cdots, y_0^{n-n_1}, 0, \cdots, 0) = x_0^i,$$

$$g_j(x_0^1, \cdots, x_0^{n_1}, y_0^1, \cdots, y_0^{n-n_1}, 0, \cdots, 0) = y_0^j,$$

and then there is an unanticipated change in ϵ from its initial value at zero. Suppose $0 < \beta < 1$ is a constant discount rate, we are interested in the induced change of a discounted, dynamic evaluation function

$$W = \sum_{t=0}^{\infty} \beta^t v(x_t^1, \cdots, x_t^{n_1}, y_t^1, \cdots, y_t^{n-n_1}).$$

We would like to find the impact of the change of ϵ on the endogenous variables, $\frac{dx_t}{d\epsilon}$, $\frac{dy_t}{d\epsilon}$, where we denote $\frac{dx_t}{d\epsilon} = (\frac{dx_t^1}{d\epsilon}, \cdots, \frac{dx_t^{n_1}}{d\epsilon})^T$, $\frac{dy_t}{d\epsilon} = (\frac{dy_t^1}{d\epsilon}, \cdots, \frac{dy_t^{n-n_1}}{d\epsilon})^T$.

For any fixed ϵ, taking differentiation on system (1) with respect to ϵ yields

$$\begin{pmatrix} \frac{dx_{t+1}}{d\epsilon} \\ \frac{dy_{t+1}}{d\epsilon} \end{pmatrix} = A \begin{pmatrix} \frac{dx_t}{d\epsilon} \\ \frac{dy_t}{d\epsilon} \end{pmatrix} + Bh_t, \tag{2}$$

where A is an $n \times n$ constant matrix, and B is an $n \times K$ constant matrix.

To rule out the exponential growth of $\frac{dx_t}{d\epsilon}$, $\frac{dy_t}{d\epsilon}$, and h_t, we assume that

Assumption 1.

1) *For any t, there exists $\bar{h} \in R^K$ and $\theta_t \in R$ such that* $|\frac{h_{t+s}}{(1+s)^{\theta_t}}| \leq \bar{h}, \forall s \geq 0.$

2) *For any t, there exists* $\begin{pmatrix} \frac{d\bar{x}_t}{d\epsilon} \\ \frac{d\bar{y}_t}{d\epsilon} \end{pmatrix} \in R^n$ *and $\sigma_t \in R$ such that*

$$-\begin{pmatrix} \frac{d\bar{x}_t}{d\epsilon} \\ \frac{d\bar{y}_t}{d\epsilon} \end{pmatrix} \leq (1+s)^{-\sigma_t} \begin{pmatrix} \frac{dx_{t+s}}{d\epsilon} \\ \frac{dy_{t+s}}{d\epsilon} \end{pmatrix} \leq \begin{pmatrix} \frac{d\bar{x}_t}{d\epsilon} \\ \frac{d\bar{y}_t}{d\epsilon} \end{pmatrix} \ \forall s \geq 0.$$

The coefficient matrix A is the Jacobian matrix of the vector function $(f_1, \cdots, f_{n_1}, g_1, \cdots, g_{n-n_1})^T$ evaluated at the steady-state values of (x^*, y^*). We suppose matrix A can be transformed into a Jordan canonical form by a transform V,

$$A = V^{-1} \Lambda V,$$

where V is an $n \times n$ matrix whose rows are linearly independent, left-eigenvectors of A. Λ is a diagonal matrix whose diagonal elements are the eigenvalues of A, which are ordered by increasing value. If the number of eigenvalues of A outside the unit circle equals the number of non-predetermined variables, then there is a unique solution for the difference equation system (2) (Blanchard and Kahn, 1980). Therefore, we make the following assumption:

Assumption 2. *A has n_1 distinct characteristic roots $\omega_1, \cdots, \omega_{n_1}$, which are on or inside the unit circle, and $n - n_1$ distinct characteristic roots $\mu_1, \cdots, \mu_{n-n_1}$, which are outside the unit circle.*

Under assumption 2, we can partition matrix Λ according to the length of x and y,

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

where Λ_1 is a $n_1 \times n_1$ diagonal matrix, its diagonal elements are the characteristic roots of A inside the unite circle, Λ_2 is a $n - n_1 \times (n - n_1)$ diagonal matrix, its diagonal elements are the characteristic roots of A outside the unite circle.

Accordingly, matrices V and B can be decomposed as follows:

$$V = \begin{pmatrix} \underset{n_1 \times n_1}{V_{11}} & \underset{n_1 \times (n-n_1)}{V_{12}} \\ \underset{(n-n_1)\times n_1}{V_{21}} & \underset{(n-n_1)\times(n-n_1)}{V_{22}} \end{pmatrix}, B = \begin{pmatrix} \underset{n_1 \times K}{B_1} \\ \underset{(n-n_1)\times K}{B_2} \end{pmatrix}.$$

In order to derive the welfare effects of exogenous policy changes of ϵ on endogenous variables, $\frac{dx_t}{d\epsilon}$, $\frac{dy_t}{d\epsilon}$, we will use the $Z-$transform. First, we recall the definition and some important and useful propositions of the $Z-$transform.

Definition: For any function f_t, let $Z(f_t,\, s) = \sum_{t=0}^{\infty} f_t s^{-t}$ be the $Z-$transform of it evaluated at s.

From the definition, we have some important and elementary propositions for the $Z-$transform:

Proposition 1. *For any two functions x_t, y_t, and any a, $b \in R$, we have*

$$Z(ax_t + by_t, s) = aZ(x_t, s) + bZ(y_t, s).$$

Proposition 2. *For any function x_t, we have*

$$Z(x_{t+1}, s) = sZ(x_t, s) - sx_0.$$

Taking the $Z-$transform of equation (2) and using the proposition 1 and 2, we arrive at

$$(sI - A)\begin{pmatrix} Z(\frac{dx_t}{d\epsilon}, s) \\ Z(\frac{dy_t}{d\epsilon}, s) \end{pmatrix} = BZ\left(h_t, s\right) + \begin{pmatrix} s\frac{dx_0}{d\epsilon} \\ s\frac{dy_0}{d\epsilon} \end{pmatrix}, \tag{3}$$

where $\frac{dx_0}{d\epsilon}$ and $\frac{dy_0}{d\epsilon}$ are the initial changes of economic variables induced by ϵ at time zero, $Z\left(h_t, s\right) = \left(Z\left(h_t^1, s\right), \cdots, Z\left(h_t^K, s\right)\right)^T$ is the $Z-$transform of function h_t.

We need to find the values of initial change $\frac{dx_0}{d\epsilon}$ and $\frac{dy_0}{d\epsilon}$ to determine $Z(\frac{dx_t}{d\epsilon},\, s)$ and $Z(\frac{dy_t}{d\epsilon},\, s)$. Because $x^1, \cdots, x^{n_1}$ are state variables, they cannot jump initially. Thus $\frac{dx_0}{d\epsilon} = 0$. To compute $\frac{dy_0}{d\epsilon}$, substituting $A = V^{-1}\Lambda V$ into equation (3) and multiplying the left side of (3) by V, we have

$$(sI - \Lambda)\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} Z(\frac{dx_t}{d\epsilon}, s) \\ Z(\frac{dy_t}{d\epsilon}, s) \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\begin{pmatrix} B_1 Z\left(h_t, s\right) \\ B_2 Z\left(h_t, s\right) + s\frac{dy_0}{d\epsilon} \end{pmatrix}. \tag{4}$$

For the eigenvalues outside the unit circle $\mu_1, \cdots, \mu_{n-n_1}$, the Z-transforms $Z(\frac{dx_t}{d\epsilon},\, \mu_1), \cdots, Z(\frac{dy_t}{d\epsilon},\, \mu_{n-n_1})$ must be finite. Because $\Lambda_2 I - \Lambda_2 = 0$ and V is invertible, we get the expression for $\frac{dy_0}{d\epsilon}$

$$\frac{dy_0}{d\epsilon} = -\left(\Lambda_2 V_{22}\right)^{-1}\left(V_{21}B_1 + V_{22}B_2\right)Z\left(h_t, \Lambda_2\right), \tag{5}$$

where $Z\left(h_t, \Lambda_2\right) = \left(Z\left(h_t, \mu_1\right), \cdots, Z\left(h_t, \mu_{n-n_1}\right)\right)$.

Combining equation (3) with equation (5), we obtain the expressions for $Z(\frac{dx_t}{d\epsilon}, s)$ and $Z(\frac{dy_t}{d\epsilon}, s)$. From which, in turn, we obtain the impact of the change of ϵ on the welfare

$$\frac{dW}{d\epsilon} = \begin{pmatrix} \bigtriangledown_x v \\ \bigtriangledown_y v \end{pmatrix}^T \sum_{t=0}^{\infty} \beta^t \begin{pmatrix} \frac{dx_t}{d\epsilon} \\ \frac{dy_t}{d\epsilon} \end{pmatrix} = \begin{pmatrix} \bigtriangledown_x v \\ \bigtriangledown_y v \end{pmatrix}^T \begin{pmatrix} Z(\frac{dx_t}{d\epsilon},1/\beta) \\ Z(\frac{dy_t}{d\epsilon},1/\beta) \end{pmatrix}, \tag{6}$$

where $Z(\frac{dx_t}{d\epsilon},1/\beta)$ and $Z(\frac{dy_t}{d\epsilon},1/\beta)$ are the $Z-$transform of $\frac{dx_t}{d\epsilon}$ and $\frac{dy_t}{d\epsilon}$ evaluated at $1/\beta$ and derived from equation (3); $\bigtriangledown_x v$ and $\bigtriangledown_y v$ are the gradient vectors of V with respect to x and y, respectively.

3 An Example

3.1 The Uzawa-Lucas model

In this section, we present the discrete-time Uzawa-Lucas model to examine the effects of fiscal-policy changes on the economy. As in Barro and Sala-i-Martin (1995), the agent is to choose the consumption path, c_t, the fraction of labor allocated to the production of physical capital, u_t, the accumulation paths for the physical capital, k_t, and human capital, h_t, to maximize the discounted utility, i.e.,

$$\max_{c_t,u_t,k_t,h_t} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{7}$$

subject to the following constraints and the given initial capital k_0, h_0

$$k_{t+1} = k_t + (1-\tau^r)r_t k_t + (1-\tau^l)\omega_t u_t h_t - c_t + \chi_t, \tag{8}$$

$$h_{t+1} = h_t + B(1-u_t)h_t, \tag{9}$$

where $0 < \beta < 1$ is the constant discount rate, $B > 0$ is a constant, r_t is the market return on the physical capital stock, ω_t is the return on the human-capital stock, u_t is the fraction of labor allocated to the production of physical capital, which is also called the working time, τ^r and τ^l are the tax rates on physical and human capital income, respectively, and finally, χ_t is the transfer from the government. $U(c_t)$ is the instantaneous utility function of the agent, which takes the CES form, i.e.,

$$U(c_t) = \{ \begin{array}{ll} \frac{c_t^{1-\sigma}}{1-\sigma}, & \sigma > 0 \text{ and } \sigma \neq 1 \\ \log c_t, & \sigma = 1 \end{array} ,$$

where $\sigma > 0$ is the elasticity of intertemporal substitution in consumption.

The first-order conditions for welfare maximization are:

$$\lambda_t = U'(c_t), \tag{10a}$$

$$\lambda_{t-1} = \beta\lambda_t[r_t(1-\tau^r)+1], \tag{10b}$$

$$\mu_{t-1} = \beta[\lambda_t\omega_t(1-\tau^l)u_t + \mu_t(B(1-u_t)+1)], \tag{10c}$$

$$B\mu_t = \lambda_t\omega_t(1-\tau^l), \tag{10d}$$

where λ_t and μ_t are the shadow prices for physical capital k_t and human capital h_t, respectively.

Suppose the output y_t is produced by the neoclassical production function with the physical capital and human capital, $y_t = f(k_t,u_t h_t)$. From the firm's profit-maximization problem, we have

$$r_t + \delta = f_k(k_t,u_t h_t), \quad \omega_t = f_{uh}(k_t,u_t h_t), \tag{11}$$

where δ is the depreciate rates of physical capital, and we suppose the depreciate rate of human capital is zero.

In order to derive an explicit solution, we specify the production function as Cobb-Douglas, $y_t = f(k_t,u_t h_t) = Ak_t^{\alpha}(u_t h_t)^{1-\alpha}$, where $A > 0$ and $0 < \alpha < 1$ are constants.

As for the government sector, suppose the government expenditure is G_t, and in the absence of government bonds, the government's budget constraint is:

$$\chi_t = \tau^r r_t k_t + \tau^l \omega_t u_t h_t - G_t. \tag{12}$$

Therefore, the dynamics of the economy can be summarized by equations (8), (9), (10), (11), and (12). Similar to Mulligan and Sala-i-Martin (1993), let $x_t = \frac{k_t}{h_t}$, $y_t = \frac{c_t}{h_t}$, and $z_t = \frac{x_t}{u_t}$. Then the dynamics of the economy can be rewritten as

$$x_{t+1} = \frac{Ax_t z_t^{\alpha} + [(1-\delta)x_t - y_t - g_t]z_t}{B(z_t - x_t) + z_t}, \tag{13a}$$

$$y_{t+1}[\beta((A\alpha z_{t+1}^{\alpha-1} - \delta)(1-\tau^r) + 1)]^{-\frac{1}{\sigma}} = \frac{z_t y_t}{B(z_t - x_t) + z_t}, \tag{13b}$$

$$\frac{z_{t+1}^{\alpha}}{(A\alpha z_{t+1}^{\alpha-1} - \delta)(1-\tau^r) + 1} = \frac{z_t^{\alpha}}{B+1}, \tag{13c}$$

where $g_t \triangleq G_t/y_t$ is the government expenditure-output ratio. For simplicity, we have assumed the transfer of the government is zero along with the balanced growth path.

On the other hand, along with the balanced growth path, the growth rates for the consumption, physical capital stock, human capital accumulation, and u_t are constants; And the ratio of government expenditure to output, g, is also constant. Therefore, along with the balanced growth path, the steady state can be derived from $x_{t+1} = x_t = x^*$, $y_{t+1} = y_t = y^*$, and $z_{t+1} = z_t = z^*$:

$$z^* = \left(\frac{B + \delta(1-\tau^r)}{A\alpha(1-\tau^r)}\right)^{\frac{1}{\alpha-1}}, \tag{14a}$$

$$x^* = \frac{1 + B - [\beta(1+B)]^{\frac{1}{\sigma}}}{B} z^*, \tag{14b}$$

$$y^* = \left[\frac{(1-g)(B + \delta(1-\tau^r))}{\alpha(1-\tau^r)} + 1 - \delta - [\beta(1+B)]^{\frac{1}{\sigma}}\right] x^*. \tag{14c}$$

In Appendix A1, we have proved that there exists that one eigenvalue inside the unit circle (denoted as λ_1), and two eigenvalues outside the unit circle (denoted as λ_2 and λ_3, respectively). Therefore, the steady state is saddle-point stable.

Now, we turn to discuss the effects of fiscal-policy changes on the economy. From the discussions above, we find that only the physical capital income tax affects the equilibrium, whereas the human capital income tax rate has no effect on the equilibrium. Suppose at time $t = 0$, the economy reaches its steady state with the capital income tax rate τ^r. Now, suppose the government announces that the capital income tax rate will be changed according to

$$\overline{\tau}_t^r = \tau^r + \epsilon h_t,$$

where h_t is a step function, ϵ is a parameter representing the intensity of the impact.

Substituting $\overline{\tau}_t^r$ into the dynamic system (10a)-(10c), and taking differentiations on system (10) with respect to ϵ evaluated at $\epsilon = 0$[1], we have

$$\begin{pmatrix} \frac{dx_{t+1}}{d\epsilon} \\ \frac{dy_{t+1}}{d\epsilon} \\ \frac{dz_{t+1}}{d\epsilon} \end{pmatrix} = \bar{A} \begin{pmatrix} \frac{dx_t}{d\epsilon} \\ \frac{dy_t}{d\epsilon} \\ \frac{dz_t}{d\epsilon} \end{pmatrix} + \bar{B} h_t, \tag{15}$$

where

$$\bar{A} = \begin{pmatrix} \frac{(Pz^*+Bx^*)D}{z^*} & -D & \frac{\alpha(1-\tau_k)+\alpha\tau^r(1+B)-M}{\alpha(1-\tau^r)} \frac{Dx^*}{z^*} \\ \frac{BDy^*}{z^*} & 1 & \frac{(\sigma-1)M-(\sigma DM-\alpha)(1+B)}{\sigma M} \frac{y^*}{z^*} \\ 0 & 0 & \frac{\alpha(1+B)}{M} \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 \\ \frac{-\alpha(1+B)By^*}{\sigma(1+B)(1-\tau^r)M} \\ -\frac{Bz^*}{(1-\tau^r)M} \end{pmatrix}$$

with $D = [\beta(1+B)]^{-\frac{1}{\sigma}}$, $M = \alpha(1+B) + (1-\alpha)(B + \delta(1-\tau^r))$, and $P = \frac{B+(\delta+\alpha(1-\delta))(1-\tau^r)}{\alpha(1-\tau^r)}$.

Taking the Z-transform of equation (15), we have

$$(sI - \bar{A}) \begin{pmatrix} Z(\frac{dx_t}{d\epsilon},s) \\ Z(\frac{dy_t}{d\epsilon},s) \\ Z(\frac{dz_t}{d\epsilon},s) \end{pmatrix} = \bar{B} Z(h_t,s) + s \begin{pmatrix} \frac{dx_0}{d\epsilon} \\ \frac{dy_0}{d\epsilon} \\ \frac{dz_0}{d\epsilon} \end{pmatrix}. \tag{16}$$

Since x_t is a state variable, it cannot jump initially, thus $\frac{dx_0}{d\epsilon} = 0$. We use the method developed in section 2.2 to calculate the values for $\frac{dy_0}{d\epsilon}$ and $\frac{dz_0}{d\epsilon}$.

Firstly, matrix $\bar{A}$ can be diagonalized as

$$\bar{A} = V^{-1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} V,$$

where V is a 3×3 matrix whose rows are linearly independent left-eigenvectors of A.

Secondly, partition matrices V and $\bar{B}$ as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where V_{11} is a real number, V_{12}, V_{21}, and V_{22} are 1×2, 2×1, 2×2 matrices, respectively. B_1 is a 1×2 matrix and B_2 is a 2×2 matrix.

Therefore, from equation (5), the initial effects of a change in the capital income tax rate on y and z can be derived as:

$$\begin{pmatrix} \frac{dy_0}{d\epsilon} \\ \frac{dz_0}{d\epsilon} \end{pmatrix} = -\left(\Lambda V_{22}\right)^{-1} \left(V_{21} B_1 + V_{22} B_2\right) Z\left(h_t,\Lambda\right), \tag{17}$$

where $\Lambda = \begin{pmatrix} \lambda_2 & \\ & \lambda_3 \end{pmatrix}$ and $Z(h_t,\Lambda) = (Z(h_t,\lambda_2), Z(h_t,\lambda_3))$.

[1] Herein the after, when taking differentiations on ϵ, we always evaluate at $\epsilon = 0$.

3.2 Initial effects on consumption and working time

Having derived the initial effects of a capital income tax rate on variables x, y, and z, we can obtain the initial effects of capital income tax rate on the initial consumption c and the initial working time u.

Because $c(0) = y(0)h(0)$, $u(0) = x(0)/z(0)$, and

$$\Delta c_0 \approx \frac{dc_0}{d\epsilon}\Delta\epsilon = \frac{dy_0}{d\epsilon}\Delta\epsilon,\ \Delta u_0 \approx \frac{du_0}{d\epsilon}\Delta\epsilon = \frac{x_0}{dz_0/d\epsilon}\Delta\epsilon.$$

We have

$$\frac{\Delta c_0}{c_0} \approx \frac{dy_0/d\epsilon}{y_0}\Delta\epsilon, \quad \frac{\Delta u_0}{u_0} \approx \frac{z_0}{dz_0/d\epsilon}\Delta\epsilon, \tag{18}$$

where $dy_0/d\epsilon$ and $dz_0/d\epsilon$ are derived in equation (17).

3.3 Effect of capital income tax on welfare

In this section, we will combine our method with the method introduced by Lucas (1990) to discuss the effect of the capital income tax rate on the social welfare.

Let

$$W(S) = \sum_{t=0}^{\infty} \beta^t U(c(t,S))$$

be the social welfare with the consumption path $c(t, S)$ under policy S.

Consider another policy S', the social welfare is $W(S')$.Let θ be a parameter satisfies

$$\sum_{t=0}^{\infty} \beta^t U((1+\theta)c(t,S)) = \sum_{t=0}^{\infty} \beta^t U(c(t,S')). \tag{19}$$

We call the number θ in units of a percentage of all consumption goods the welfare gain of a change in policy from S to S'.

Because $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, we have $\sum_{t=0}^{\infty} \beta^t U((1+\theta)c(t,S))= (1+\theta)^{1-\sigma} W(S)$ and equation (19) can be reduced to

$$(1-\sigma)\log(1+\theta) = \log\frac{W(S')}{W(S)} = \log(1+\frac{\Delta W}{W}).$$

Therefore[2],

$$\theta \approx \frac{1}{1-\sigma}\frac{dW}{W d\epsilon}\Delta\epsilon.$$

and from the technical details in Appendix A2, we have

$$\frac{dW}{W d\epsilon} = G\left[Z(\frac{dy_t}{d\epsilon},\bar{\beta}\gamma^{1-\sigma}) + \frac{By^*}{z^*(\bar{\beta}\gamma^{-\sigma}-\gamma)}\left[\frac{x^*}{z^*}Z(\frac{dz_t}{d\epsilon},\bar{\beta}\gamma^{1-\sigma}) - Z(\frac{dx_t}{d\epsilon},\bar{\beta}\gamma^{1-\sigma})\right]\right], \tag{20}$$

where $\bar{\beta} = 1/\beta$ and $G = \frac{(1-\sigma)(\bar{\beta}\gamma^{1-\sigma}-1)}{y^*\bar{\beta}\gamma^{1-\sigma}}$.

Equations (18) and (20) give the impacts of a change of the capital income tax rate on initial consumption c, initial working hours u, and the social welfare W. In the next section, we will use numerical method to calculate these effects.

[2] We have used $\log(1+x) \simeq x$.

3.4 Numerical solution

In order to compare our solution with those presented in Jones, Manuelli and Rossi (1993) and Ortigueria (1998), we present our own numerical results. Similar to Jones, Manuelli and Rossi (1993), we specify the parameters as $\alpha = 0.36$, $\beta = 0.98$, $\delta = 0.1$, which are also estimated from the many existing time-series observations. We also specify the experimental parameters $\sigma = 1.5$, $A = 0.37$, and $B = 1$.

Suppose $h_t = \{ \begin{array}{ll} 1, & 0 \leq t \leq T \\ 0, & t > T \end{array}$. Then for $s > 1$, we have $Z(h_t,\, s) = \frac{s^{T+1}-1}{s^T(s-1)}$. Therefore, the policy change is permanent when $T \to \infty$ and $h_t \equiv 1$, $Z(h_t,\, s) = \frac{s}{s-1}$. In the following simulation, we let T change from 2, 5, 10, 20, to ∞ and present the numerical solution in Table 1.

Table 1: Effects of capital income tax on economy

		$T = 2$	$T = 5$	$T = 10$	$T = 20$	$T = \infty$
$\tau^r = .05$	$\frac{\Delta c}{c}$	-2.649e-3	-7.627e-4	3.99e-3	1.131e-2	2.106e-2
	$\frac{\Delta u}{u}$	-1.0909e-1	-7.1764e-2	-5.3566e-2	-4.1384e-2	-3.2031e-2
	θ	2.7320e-5	7.2995e-5	1.5829e-4	3.1242e-4	6.0026e-4
$\tau^r = .10$	$\frac{\Delta c}{c}$	-5.1878e-3	-8.0457e-4	9.5724e-3	2.5231e-2	4.5856e-2
	$\frac{\Delta u}{u}$	-2.0885e-1	-1.3789e-1	-1.0298e-1	-7.9453e-2	-6.1516e-2
	θ	5.9189e-5	1.5716e-4	3.3879e-4	6.6529e-4	1.3117e-3
$\tau^r = .15$	$\frac{\Delta c}{c}$	-7.5381e-3	9.0640e-5	1.7126e-2	4.2340e-2	7.5150e-2
	$\frac{\Delta u}{u}$	-2.9933e-1	-1.9834e-1	-1.4817e-1	-1.1412e-1	-8.8375e-2
	θ	9.6620e-5	2.5481e-4	5.4588e-4	1.0664e-3	2.0883e-3
$\tau^r = .20$	$\frac{\Delta c}{c}$	-9.5929e-3	2.2030e-3	2.7131e-2	6.3341e-2	1.0991e-1
	$\frac{\Delta u}{u}$	-3.8055e-1	-2.5308e-1	-1.8903e-1	-1.4531e-1	-1.1252e-1
	θ	1.4091e-4	3.6891e-4	7.8511e-4	1.5254e-3	2.9669e-3
$\tau^r = .25$	$\frac{\Delta c}{c}$	-1.1205e-2	5.8987e-3	4.0204e-2	8.9143e-2	1.5136e-1
	$\frac{\Delta u}{u}$	-4.5254e-1	-3.0201e-1	-2.2544e-1	-1.7289e-1	-1.3386e-1
	θ	1.9377e-4	5.0327e-4	1.0636e-3	2.0549e-3	3.9691e-3
$\tau^r = .30$	$\frac{\Delta c}{c}$	-1.2171e-2	1.1662e-2	5.7145e-2	1.2093e-1	2.0110e-1
	$\frac{\Delta u}{u}$	-5.1529e-1	-3.4504e-1	-2.5726e-1	-1.9675e-1	-1.5228e-1
	θ	2.5741e-4	6.6286e-4	1.3905e-3	2.6709e-3	5.1227e-3

Therefore, with an increase in the capital income tax rate, initial working hours will decrease. This relationship is consistent with the one in Ortigueira (1998). However, the magnitude of the effect in our paper is much less than that of Ortigueira (1998). For example, Ortigueira (1998) presents that the effect of capital income tax rate on initial working hours is about -0.3441 when $\tau^r = 0.30$. However, it is only -0.15228 here, which is about 50% percent of Ortigueira (1998)'s. Table 1 also illustrates that the effect of a capital income tax on initial working hours is decreasing with the persistence of the policy change, the effect decreases from -0.515 to -0.15 when T increases from 2 to ∞.

A change in the capital income tax rate impacts on the initial consumption. This is consistent with Ortigueira (1998) and Jones, Manuelli and Rossi (1993), however, the magnitude of effect is larger than that of Ortigueira (1998): the effect of capital income tax rate on initial consumption is about 0.14 when $\tau^r = 0.30$, and, however, it is only 0.201 in our paper.

Table 1 further presents the welfare cost of capital income tax rate, from which we know that an increase in the capital income tax rate increases the welfare cost of capital income tax rate. Comparing with that of Ortigueira (1998), the magnitude of welfare cost is less. For example, the welfare cost is 0.0026 when $\tau^r = 0.3$ in our numerable illustration, which is two times larger than that the one in Ortigueira (1998).

4 Concluding Remarks

This paper presents a method to study the effects of policy changes on the economy in the discrete-time case. Different from those methods presents by Judd (1982) and Meijdam and Verhoeven (1998), the method developed here has no constraint on the dimension of the system. And the method developed here not only examines the effects of permanent policy change, but also takes care the impact of temporal policy change on the social welfare.

5 Appendices

A1: The Stability of Steady State

Because $\bar{A}$ can be diagonalized, λ_2 and λ_3 are the eigenvalues of the following matrix:

$$\begin{pmatrix} (P + B\frac{x^*}{z^*})D & -D \\ BDy^*/z^* & 1 \end{pmatrix}.$$

It can be rewritten as:

$$\begin{pmatrix} (P+1+B)D-1 & -D \\ (P-D^{-1})((1+B)D-1) & 1 \end{pmatrix}.$$

Therefore, λ_2 and λ_3 satisfy

$$\lambda_2 + \lambda_3 = (P+1+B)D,$$

$$\lambda_2\lambda_3 = (P+1+B)D - 1 + (PD-1)((1+B)D-1).$$

Without loss of generality, we can assume $1 < \lambda_2 \leq \lambda_3$, so we get:

$$\lambda_2 = \min\{(1+B)D, PD\}, \lambda_3 = \max\{(1+B)D, PD\}.$$

Therefore $\lambda_2 > 1$ and $\lambda_3 > 1$ when $D^{-1} < \min\{P, 1+B\}$. Because $\lambda_1 = \frac{\alpha(1+B)}{\alpha(1+B)+(1-\alpha)(B+\delta(1-\tau_k))} < 1$, we know that the steady state is saddle-point stable.

A2: Computation of $dW/d\epsilon$

From equations (16) and (17), for every $s > 1$, we can compute $Z(\frac{dx_t}{d\epsilon}, s)$, $Z(\frac{dy_t}{d\epsilon}, s)$, and $Z(\frac{dz_t}{d\epsilon}, s)$. Then, we can compute the impact of social welfare, we have

$$\frac{dU(c_t)}{d\epsilon} = \frac{dU(y_t h_t)}{d\epsilon} = y^{*-\sigma} h_t^{-\sigma}(h_t \frac{dy_t}{d\epsilon} + y^* \frac{dh_t}{d\epsilon}). \quad \text{(A1)}$$

Form $h_{t+1} = [B(1-u_t)+1]h_t$, we have

$$\frac{dh_{t+1}}{d\epsilon} = [B(1-u^*)+1]\frac{dh_t}{d\epsilon} - Bh_t\frac{du_t}{d\epsilon}. \tag{A2}$$

And from the definition $u_t = \frac{x_t}{z_t}$, we have

$$\frac{du_t}{d\epsilon} = \frac{1}{z^*}\frac{dx_t}{d\epsilon} - \frac{x^*}{z^{*2}}\frac{dz_t}{d\epsilon}. \tag{A3}$$

Substituting equation (A3) into equation (A2) and reminding that $h_t = h_0[\beta(1+B)]^{\frac{t}{\sigma}}$, we have

$$\frac{dh_{t+1}}{d\epsilon} = [B(1-u^*)+1]\frac{dh_t}{d\epsilon} - \frac{Bh_0}{z^*}\gamma^{-t}\frac{dx_t}{d\epsilon} + \frac{Bh_0x^*}{z^{*2}}\gamma^{-t}\frac{dz_t}{d\epsilon}, \tag{A4}$$

where $\gamma = [\beta(1+B)]^{-\frac{1}{\sigma}}$.

For any $s > \frac{1}{\gamma}$, taking Z-transform on equation (A4), we obtain

$$Z(\frac{dh_t}{d\epsilon},s) = -\frac{Bh_0}{z^*(s-\gamma)}Z(\frac{dx_t}{d\epsilon},\gamma s) + \frac{Bh_0x^*}{z^{*2}(s-\gamma)}Z(\frac{dz_t}{d\epsilon},\gamma s). \tag{A5}$$

Therefore, we have

$$\begin{aligned}
\frac{dW}{d\epsilon} &= \frac{dZ(U(c_t),\bar\beta)}{d\epsilon} = Z(\frac{dU(c_t)}{d\epsilon},\bar\beta) \\
&= y^{*-\sigma}h_0^{-\sigma}\left[h_0 Z(\frac{dy_t}{d\epsilon},\bar\beta\gamma^{1-\sigma}) + y^*Z(\frac{dh_t}{d\epsilon},\bar\beta\gamma^{-\sigma})\right] \\
&= y^{*-\sigma}h_0^{1-\sigma}\left[Z(\frac{dy_t}{d\epsilon},\bar\beta\gamma^{1-\sigma}) + \frac{By^*}{z^*(\bar\beta\gamma^{-\sigma}-\gamma)}\left[\frac{x^*}{z^*}Z(\frac{dz_t}{d\epsilon},\bar\beta\gamma^{1-\sigma}) - Z(\frac{dx_t}{d\epsilon},\bar\beta\gamma^{1-\sigma})\right]\right],
\end{aligned}$$

where $\bar\beta = 1/\beta$.

Divided by $W = \frac{\gamma^{1-\sigma}\bar\beta y^{*1-\sigma}h_0^{1-\sigma}}{(1-\sigma)(\gamma^{1-\sigma}\bar\beta-1)}$, we get equation (20) in the text.

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