Power Dynamics

——*Multiple Equilibria, Cyclical Fluctuations, and Local Stability in Intertemporal General Equilibrium Models*

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Abstract

Based on the theory of power proposed by John Galbraith, Bertrand Russell and other social scientists, this paper offers two positive intertemporal general equilibrium models to understand: i.) How do rational people decide their optimal consumption, property accumulation, and power? ii.)What causes people to choose different growth path of power? iii.)Why would two people, whose power and wealth endowment levels are quite close, differ so drastically in their future practice? iv.) Why could the rational pursue of power in a "perfect" world is compatible with cyclical power patterns? The simple two-dimensional model considers the dynamic property of power. The second model is an extension of the first model, and analyses the wealth effect on power accumulation. Numerical simulations have provided strong support for our modeling approaches.

Key words: Power; Stable limit cycles; Hopf bifurcation; Multiple equilibria.

1. Introduction

Not many get through a conversation without a reference to power. Max Weber, the German sociologist and political scientist (1864-1920), in '*Max Weber on law in economy and Society*' defines: power is "the possibility of imposing one's will upon the behavior of other persons". Thus, corporations and trade unions are said to be powerful, and multinational corporations dangerously so; Newspaper publishers, the heads of the broadcasting networks, and the more articulate, uninhibited, intelligent, or notorious of their editors, columnists, and commentators are the powers that be. To take it universally, reputation of power is power; what quality so ever make a man beloved or feared of many, or the reputation of such quality, is power; good success is power; nobility is power; eloquence is power; and sciences are also power. Bertrand Russell (1938) was led to the thought that power, along with glory, remains the highest aspiration and greatest reward of humankind.

Power in the entire range of the social sciences is important, but at the same time it is so seriously neglected, especially in economics. In '*Power: A New Social Analysis*', Russell states

> 'The orthodox economists, as well as Marx … were mistaken in supposing that economic self-interest could be taken as the fundamental motive in the social sciences. The desire for commodities, when separated from power and glory, is finite, and can be fully satisfied by a moderate competence. The really expensive desires are not dictated by a love of material comfort. … When a moderate degree of comfort is assured, both individuals and communities will pursue power rather than wealth: they may seek wealth as a means to power, or they may forgo an increased of wealth as a means to powers, or they may forgo an increase of wealth in order to secure an increase of power, but the former case as in the latter their fundamental motive is not economic. … This error in orthodox and Marxist economics … has caused some of principal events of recent times to be misunderstood. It is only by realizing that love of power is the cause of the activities that are important in social affairs that history, whether ancient or modern, can be rightly interpreted' (p. 12).

Russell is occupied by the role of power in social sciences and he declares that power is the fundamental concept in social science, in the same sense in which energy is the fundamental concept in physics:

> 'The laws of social dynamics are laws which can only be stated in terms of power. …Love of power, therefore, is a characteristic of men who are causally important. We should, of course, be mistaken if we regarded it as the sole human motive, but this mistake would not lead us to much astray as might be expected in the search for causal laws in social science, since love of power is the chief motive producing the changes which social science has to study. … The laws of social dynamics are--so I shall contend--only capable of being stated in terms of power in its various forms' (pp. 13-15).

Traditional researches on power such as Bertrand Russell (1938), John Kenneth Galbraith (1983), and Max Weber (1954) have mainly worked from the view of sociology and political science, not from economics. Unmentioned in nearly all references to it is the highly interesting questions as to how do the rational people decide their optimal consumption, property accumulation, and power? What cause people to choose different power growth path? Why would two persons, whose power and wealth endowment levels are quite close, differ so drastically in their future practices? And why could the rational pursue of power in a "perfect" world is compatible with cyclical power patterns? It is these questions that this paper addresses.

Here our model is based on the theory of power proposed by Galbraith, Russell and other social scientists. We define the preference function on both consumption and power, which reflects the essence of the love of power: power is pursued not only for the service it render to personal interests, values, or social perceptions but also for its own sake, for the emotional and material rewards inherent in its possession and exercise (see Galbraith (1983), p.10.). In Section 2, we set up a very simple two-dimensional model, which only considers the power accumulation with the initial power endowment. As a result of the presence of the so-called power effects in the preference function, there exist multiple equilibria: some point is saddle equilibria and some are totally unstable equilibria. Therefore, people with the same preference and the same time discount rate may have different steady states and different growth paths depending on the difference in their initial endowment of power.

Following the ideas of Galbraith (1983) on the sources of power, we offer an alternative intertemporal general equilibrium model to capture the wealth effect on power in Section 3. As in the simple two-dimensional model, we can also get the existence of multiple equilibria, which imply that two persons with quite close initial power and wealth endowment may consume and allocate time between the pursuit of power and the production of wealth on completely different growth paths; or they would converge to different steady states. Under the framework of both wealth and power accumulation, we can show that if the parameter that measures the degree of desire (or love) for power rises above some critical values, limit cycles emerge around the stationary point. The stable limit cycles characterize and explain human cyclical patterns of power and wealth accumulation. It is shown that this cyclical pattern is even optimal. We conclude this paper with a few remarks in Section 4.

2. The Simple Two-dimensional Model

2.1. The preference on power

We assume that the utility function, $U(c, p)$, is defined on consumption, c, and power, p. Further, the utility function satisfies the following properties:

$$U_c \geq 0,\ U_{cc} \leq 0,\ U_p \geq 0,\ U_{pp} \leq 0,\ U_{cp} \geq 0$$

The definition of this utility function reflects the reality that power is pursued not only for the service it render to personal interests, values, or social perceptions but also for its own sake, for the emotional and material rewards inherent in its possession and exercise (see Galbraith (1983), p.10.). In William Hazlitt's words, "The love of power is the love of ourselves."

The purpose of pursuing power is the exercise of power itself, rather than only for the material reward that it can serve to bring. This idea has been also taken by Thomas Hobbes (1651), Bertrand Russell (1938), and John Kenneth Galbraith (1983), among many others.

In the '*Leviathan*', Hobbes makes the following description of men's desire for power:

> 'Nor can a man any more live whose desires are at an end than he whose senses and imaginations are at a stand. Felicity is a continual progress of the desire from one object to another, the attaining of the former being still but the way to the latter. The cause whereof is that the object of man's desire is not to enjoy once only, and for one instant of time, but to assure forever the way of his future desire. … *So that in the first place, I put for a general inclination of all mankind a perpetual and restless desire of power after power, that cease only in death.*' (*The Great Books of the Western World,* Vol. 23, p. 76, italic added).

Bertrand Russell (1938) also develops the same idea in his book ‘*Power: A New Social Analysis* ’. He says that

> ‘Love of power, like lust, is such a strong motive that it influences most men’s actions more than they think it should. … We must admit that men have acted badly from love of power, and will continue to do so; but we ought not, on this account, to maintain that love of power is undesirable in forms and circumstances in which we believe it to be beneficial or at least innocuous.’ (p. 266).

Russell continues to describe the motive for pursing power:

> ‘Of the infinite desires of man, the chief is the desires for power and glory. These are not identical, though closely allied: the Prime Minister has more power than glory, the King has more glory than power. As a rule, however, the easiest way to obtain glory is to obtain power; this is especially the case as regards the man who are active in relation to public events. The desire for glory, therefore, prompts, in the main, the same actions as are prompted by the desire for power, and the two motives may, for most practical purposes, be regarded as one.’(pp. 11-12). ‘Those whose love of power is not strong are unlikely to have much influence on the course of events. The men who cause social changes are, as a rule, men who strongly desire to do so. Love of power, therefore, is a characteristic of men who are causally important. *We should, of course, be mistaken if we regarded it as the sole human motive, but this mistake would not lead us to much astray as might be expected in the search for causal laws in social science, since love of power is the chief motive producing the changes which social science has to study.’* (pp. 14-15, italic added).

Galbraith (1983) also shares this view. He regards the instinctive motive for pursing power as its own sake:

> ‘[T]he purposes for which power is being sought will often be extensively and thoughtfully hidden by artful misstatement. The politician who seeks office on behalf of the pecuniary interests of affluent supporter will be especially eloquent in describing himself as a public benefactor, even a diligent and devoted friend or the poor. … *Much less appreciated is the extent to which the purpose of power is the exercise of power itself.* In all societies, from the most primitive to the ostensibly most civilized, the exercise of power is profoundly enjoyed. Elaborate rituals of obeisance admiring multitudes, applauded speeches, precedence at dinner and banquets, applauded speeches, precedence at dinners and banquets, a place in the motorcade, access to the corporate jet, the military salute-celebrate the possession of power. These rituals are greatly rewarding; so are the plea and intercessions of those who seek to influence others in the exercise of power; and so, of course, are the acts of exercises--the instructions to subordinates, the military commands, the conveying of court decisions, the statement at the end of the meeting when the person in charge says, “Well, this is what we’ll do.”’ (*The Anatomy of power,* pp. 9-10, italic added).

At the same time, Galbraith admits:

> ‘However, that power is thus wanted for its own sake cannot, as a matter of basic decency, be too flagrantly conceded. It is accepted that an individual can seek power to impose his moral values on others, or to further a vision of social virtue, or to make money. And, as noted, it is permissible to disguise one purpose with another … Yet while the pursuit of power for the sake of power cannot be admitted, the reality is, as ever, part of the public consciousness. Politicians are frequently described as “power-hungry”; the obvious implication is that they seek power to satisfy an appetite. Corporations take over other corporations not in pursuit of profits but in pursuit of the power that goes with the direction of a yet large enterprise. This, too, is recognized. American politicians--senators, congressmen, cabinet officers, and Presidents-regularly sacrifice wealth, leisure, and much else to the rigors of public office. *That the nonspecific exercise of power and the access to its rituals are part of the reason is fairly evident. Perhaps only from those so rewarded are the pleasures of power for its own sake extensively concealed.’* (pp. 10-11, italic added).

The references can continue forever. Dr. Harvey Rich, a Washington, D.C., psychoanalyst, says: “The healthy individual who gains power loves it.’ (Quoted in the *New York Times,* November 9, 1982). Bertrand de Jouvenel puts the matter more vividly: “The leader of any group of man ... feels thereby an almost physical enlargement himself ... Command is a mountain top. The air breathed there is different, and the perspectives seen there are different, from those of the valley of obedience.” (*On Power: Its Nature and the History of Its Growth* [New York: Viking Press, 1994], p. 116). John F. Kennedy, a man of some candor in public expression, expresses his purpose to seek power merely for the very great enjoyment that it accords. “I run for president,” he said, “because that is where the action is.” By *action* he was close to meaning power. Nietzsche endows his superman the lust for power to extreme. His “noble” man is a being wholly devoid of sympathy, ruthless, cunning, cruel, concerned only with his own power. The nutshell of Friedrich Nietzsche’s philosophy is like what King Lear, on the verge of madness, says:

I will do such things—
What they are get I know not—but they shall be
The terror of the earth.

2.2. The budget constraint and the dynamic equations of power

Suppose total time and energy the representative agent have at time t to be $T(t)$, which is simply normalized to be one unit. Let

(2.1) $$T_1(t) + T_2(t) = 1$$

where $T_1(t)$ is the time allocated to work and earn money, and $T_2(t)$ is the time for political activities, group fighting, competing, and publicity-seeking, among many other things.

The budget constraint for the agent is specified as follows:

(2.2) $$c(t) = T_1(t) + Rp(t)$$

where we have assumed that the wage rate is to be 1; $Rp(t)$ is the material gains from power.[1]

The level of power is described by the dynamical equation:

(2.3) $$\frac{dp}{dt} = F(T_2(t), p) - \delta p(t)$$

where $F(T_2(t),p)$ is power generation function, which measures the effectiveness of organization, political activity, and status qua in power; δ is power’s depreciation rate. Furthermore, we assume that F is typically neoclassical:

$$F_1 \geq 0,\ F_2 \geq 0,\ F_{12} \geq 0,\ F_{11} \leq 0,\ F_{22} \leq 0\,;$$

Substituting $T_2(t)$ from equations (2.1) and (2.2), we can define a new function

$$f(c,p) \equiv F(1-c+Rp, p)\,.$$

then,

$$f_1 = -F_1 < 0;\ f_{11} = F_{11} < 0;\ f_2 = F_1 R + F_2 > 0;$$

$$f_{12} = -F_{11}R - F_{12};\ f_{22} = F_{11}R^2 + 2F_{11}R + F_{22} < 0$$

the sign of f_{12} is not determined. The dynamic equation of power is changed to

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[1] Power can bring about revenue, because it has communicability. By the rent-seeking theory of Public Choice School, R can be regarded as the rent rate of power.

(2.3')

$$\frac{dp}{dt} = f(c,p) - \delta p$$

In the rest of this subsection, we are going to reason the theoretical basis of organization structure and power's depreciate rate δ in (2.3).

Berle (1969) states: 'No collective category, no class, no group of any kind in and of itself wields power or can use it. Another factor must be present: that of organization.'[2] Russell (1938) thinks: 'Power is dependent upon organizations in the main, but not wholly. Purely psychological power, such as that of Plato or Galileo, may exit without any corresponding social institution. But as a rule even such power is not important unless it is propagated by a Church, a political party, or some analogous social organism.'(p. 158). Some scholars, among them Charles E. Lindlom, even hold that organization, including that manifested in government, is the ultimate source of all power.[3]

Therefore, defining the power generation function is a way to model the nature of organization. In this respect, for space limitation, we add only one more excellent passage from Galbraith (1983). Galbraith characters three sources of power: personality, property (which includes disposable income), and organization. He analyses:

> 'Organization, the most important source of power in modern societies, has its foremost relationship with conditioned power.[4] It is taken for granted that when an exercise of power is sought or needed, organization is required. From the organization, then, come the requisite persuasion and the resulting submission to the purposes of the organization.(p. 6). … There is a case here: property and personality have effect only with the support of organization' (p. 54).

That power is subject to a depreciation rate reflects idea of the countervailing power as *the dialectic of power* in Galbraith (1983). Galbraith contents that as so often happens in the exercise of power, the resort to countervailing power is automotive:

> 'So far our concern has been with how power is exercise and extended, but we must also understand how it is resisted, for this resistance is as integral a part of the phenomenon of power as its exercise itself. … In fact, modern society is in equilibrium, more or less, between those who exercise power and those who counter it (p. 72). … We may lay it down as a rule that almost any manifestation of power will induce am opposite, though not necessarily equal, manifestation of power. Any effort to bend people to the will of others will encounter in some form an effort to resist that submission. … The power originating in personality is ordinarily answered by a strong personality; that originating in property is meet by property; that having its origins in organization is normally countered by organization' (pp. 74-75).

2.3. The dynamics of the model and the properties of the equilibrium

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[2] Adolf A. Berle, Jr., *Power* (New York: Harcourt, Brace and World, 1969), p.63.

[3] Charles E. Lindblom, *Politics and Markets: The World political-Economic Systems* (New York: Basic Books, 1977), p. 26.

[4] In '*The Anatomy of Power',* Galbraith describes condign, compensatory, and conditioned power as three instruments for wielding or enforcing power. He contented himself with a definition close to everday understanding: Condign power wins submission by the ability to impose an alternative to the preferences of the individual or group that is sufficiently unpleasant or painful so that these preferences are abandoned. Compensatory power wins submission by the offer of affirmative reward—by the giving of something of value to the individual so submitting. Conditioned power is exercise by changing belief.

The agent's objective is to maximize a discounted stream of utility over an infinite horizon with a positive time discount rate ρ:

(2.4) $$\int_0^{\infty} U(c,p)\exp(-\rho t)dt$$

subject to constraints (2.3'). The initial power level is given by $p(0)$, which can be regarded as personality and personal charisma.[5]

The current value Hamiltonian H is defined by

$$H = U(c,p) + \lambda[f(c,p) - \delta\, p]$$

where λ is the costate variable of eq. (2.3').

The necessary conditions that optimal consumption and power must satisfied are described in the following equations:

(2.5) $$\dot{c} = \frac{f_1}{U_{cc}f_1 - U_c f_{11}}\left[U_c(\rho+\delta-f_2) + U_p f_1 + \frac{U_c f_{12} - U_{cp} f_1}{f_1}(f-\delta\, p)\right]$$

(2.6) $$\dot{p} = f - \delta\, p$$

(2.7) $$\lim_{t\to\infty} \lambda\, p \exp(-\rho t) = 0$$

It is interesting to note that it is possible for this dynamic system that the stationary point is not unique, and a stationary point will not have the saddle point property. To see this, denote the equilibrium values of consumption and power as c^* and p^*, then

(2.8) $$U_c{}^*(\rho+\delta-f_2{}^*) + U_p{}^* f_1{}^* = 0$$

(2.9) $$f^* = \delta\, p^*$$

Linearizing this system around its steady state, and evaluating all derivatives at the steady state, gives

(2.10) $$\begin{bmatrix}\dot{c}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}\varphi G & \varphi H\\ f_1{}^* & f_2{}^* - \delta\end{bmatrix}\begin{bmatrix}c-c^*\\ p-p^*\end{bmatrix}$$

where

$$\varphi = \frac{f_1{}^*}{U_{cc}{}^* f_1{}^* - U_c{}^* f_{11}{}^*} \le 0;$$

$$G = U_p{}^* f_{11}{}^* + U_{cc}{}^*(\rho+\delta-f_2{}^*) \le 0;$$

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[5] Galbraith (1983) classifies personality as one of three sources of political power. He states: 'personality—leadership in the common reference—is the quality of physique, mind, speech, moral certainty, or other personal trait that gives access to one or more of the instruments of power (p. 6). … The effective personality wins submission by persuasion—by cultivating belief, by "exercising leadership." Which specific aspects of personality give access to conditioned power are among the most discussed questions of our time and, indeed, of all time. … At a most commonplace level, mental resource, precision, and acuity, charm, seeming honesty, humor, solemnity, and much more can be important. So also the ability to express thought in cogent, eloquent, repetitive, or otherwise compelling terms.' (p. 40).

$$H = U_{cp} * [\rho + 2(\delta - f_2 *)] + U_{pp} * f_1 * + U_p * f_{12} * - U_c * f_{22} * + \frac{U_c * f_{12} *}{f_1 *}(f_2 * - \delta),$$

here the sign of H is undetermined.

Calculating the trace of the Jacobian matrix of eq. (2.10), yields:

(2.11) $$\varphi G + f_2 * - \delta = \rho$$

as the trace is to the sum of the two characteristic roots of the system, at least one of the roots is positive. Therefore, we cannot have a stable equilibrium point.

Next, the determinant of the matrix is

(2.12) $$\Delta = \varphi\left[G(f_2 * - \delta) - Hf_1 *\right]$$

It is easy to see that the sign of Δ is undetermined. For Δ is the product of two characteristic roots, negative Δ implies that one root is positive and one negative, i.e. the equilibrium point is saddle-point stable. If Δ is positive, then both roots will be positive as the existence of two negative toots contradicts (2.11), i.e. the equilibrium point is totally unstable. Since the complexity of necessary conditions for the existence of multiple equilibria and the stability of the equilibrium points, in the following subsection, we will present some numerical examples.

2.4. Numerical examples

The existence and property of multiple equilibria are more complicated in our model, since consumption, *c,* depends also on power, *p,* by the agent's budget constraint eq. (2.2). We first present an example that gives a unique saddle-point equilibrium and a unique optimal path.

Example 1. As in Heng-fu Zou (1991, 1994), we assume that the utility function is separable in consumption and power:

$$U(c, p) = \ln c + \pi \ln p$$

where π measures the degree of desire for power.

The power production function is standard Cobb-Douglas form:

$$f(c, p) = A(1 - c + Rp)^{\alpha} p^{\beta} \quad (\alpha > 0, \beta > 0, \alpha + \beta \le 1)$$

Let

$$\alpha = 0.3 \;\; \beta = 0.7 \;\; A = 0.8 \;\; \delta = 0.5 \;\; \rho = 0.6 \;\; \pi = 0.2 \;\; R = 0.5.$$

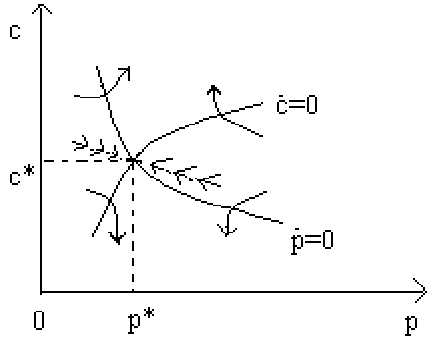

Figure 1. Unique saddle equilibrium

The steady-state eqs. (2.8)-(2.9) yield a unique equilibrium point, $(c^*, p^*) = (1.12, 0.412002)$. Substituting these values into the determinant of the steady-state matrix, eq. (2.12), we get $\Delta = -0.309923$. This is the familiar case of "saddle path stability". Figure 1 illustrates this case; the optimal path is broken line that goes through it. There is a one-dimensional manifold in {*c, p*} space with the property: trajectories that begin on this manifold converge to the steady state, but all other trajectories diverge. In this case for every p_0 in the neighborhood of the p^* there will exist a unique c_0 in the neighborhood of c^* that generates a trajectory converging to {*p*, c**}, therefore the equilibrium will be unique in the neighborhood of the steady state.

The unique saddle equilibrium implies that all the people with the same preferences and at the same society structure will eventually reach the same power and consumption level, no matter how differences among their initial power endowment.

Second, we present an example that gives a unique totally unstable equilibrium.

Example 2. Assume the same power generation function as in *Example 1.* Let

$$\alpha = 0.2 \quad \beta = 0.6 \quad A = 2 \quad \delta = 0.5 \quad \rho = 0.6 \quad \pi = 1 \quad R = 0.55 .$$

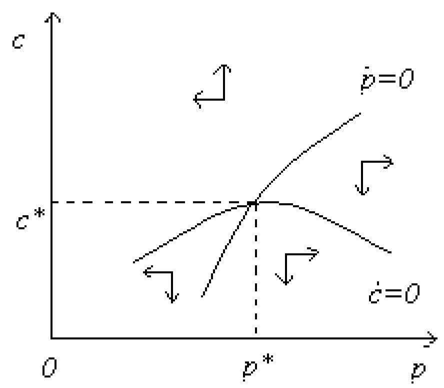

Figure 2. A unique totally unstable equilibrium

This case yields a unique equilibrium in the positive-half space {*c, p*}: $(c^*, p^*) = (58.944, 140.306)$; the determinant of the matrix $\Delta = 27.8509$. Therefore, there is a stable manifold of dimension zero (an unstable steady state). The property of trajectory is illustrated in Figure 2. All the trajectories diverge from the steady state, but the eventual fate cannot be determined from the properties of the Jacobian evaluated at the steady state. They may eventually violate nonnegativity constraints or transversality condition, or they may settle down to a limit cycles or to some more complicated attracting set; we can not therefore invoke the negative Bendixon criterion (see Guckenheirmer and Homes (p. 44, Theorem 1.8.2)) to rule out limit cycles.

Third, we present an example that gives two equilibria: one is saddle point one totally unstable point.

Example 3. Assume that the form of the power generation function is also the same as that of *Example 1*. The utility function changes to

$$U(c, p) = \ln c + \pi \ln p + \tau p$$

Let

$$\alpha = 0.7 \quad \beta = 0.3 \quad A = 0.1 \quad \delta = 0.06155722$$

$$\rho = 0.65 \quad \pi = 2 \quad R = 0.6 \quad \tau = 1$$

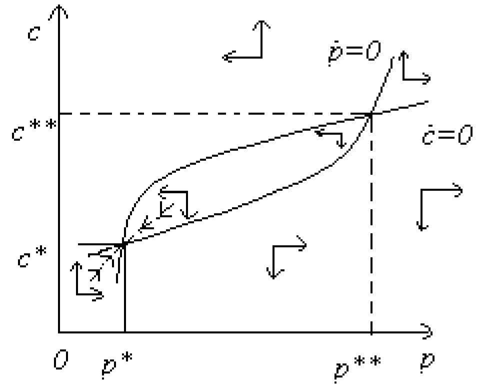

Figure 3. Two equilibria

This case yields two equilibria in the positive-half space $\{c, p\}$: $(c^*, p^*) = (1.03221, 0.322054)$, $(c^{**}, p^{**}) = (7.21014, 62.1014)$; the determinant of the matrix $\Delta_1 = -0.146977$, $\Delta_2 = 0.00533028$. Therefore, the equilibrium point (c^*, p^*) is a saddle point, and (c^{**}, p^{**}) is a totally unstable point. The property of trajectory is illustrated in Figure 3. Note that the equilibria of power have important "critical' meanings. The theory says that if initially the system start at a point $p(0)$, smaller than (or shift from) p_2^* or smaller than (or shift from) p_1^*, then it is optimal to develop p_1^*; whilst in the opposite case (if the initial situation $p(0)$ is above or shift from p_2^*), no equilibrium point is found, so that the system becomes unstable. If $p(0) = p_2^*$, then it is optimal to develop to stay there!

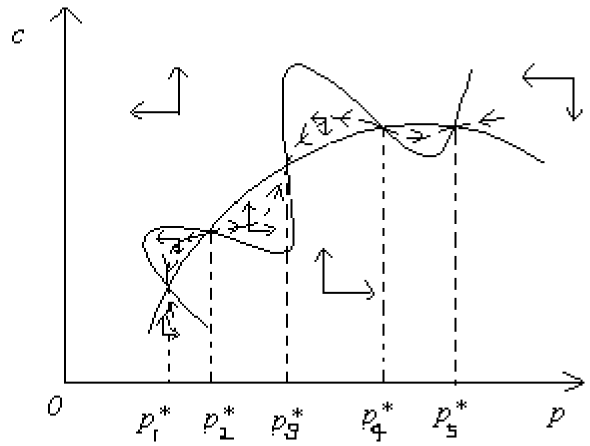

Figure 4. Multiple equilibria

It is clear that this model can lead to any number of such stationary points. Figure 4 plots five stationary points, and the optimal path is the broken line that goes through them. As we have seen, the simpler two-dimensional model illustrates that power effect could lead to multiply equilibria, therefore, people with the same preference and the same time discount rate may have different steady states and different growth path depending on the difference in their power effects. Note that if there is a large enough number of such stationary points, the optimal strategy of power growth will justify only narrow ranges of power development.

3. The Model with Wealth Accumulation

At the previous section, we gave a simpler model that only consider one state variable—power; In this section, we modify the model by introducing another state variable—wealth, and analyze the dynamic property of the wealth effect on power.

3.1. Power effect of wealth

Wealth brings about power. Lord Acton (1988) contends that: 'Power goes with property.'[6] Galbraith (1983), following the long tradition of sociology and political science, classifies wealth as one of three sources of power (personality, property and organization). He says that:

> 'Of the three sources of power, property is seemingly the most forthright. Its possession gives access to the most commonplace exercise of power, which is the will of one person to another by straightforward purchase. The employer thus bends works to his purpose, the man of wealth his chauffeur, the special-interest group its kept politicians, the lecher his mistress. (p. 47). … Property or wealth accords an aspect of authority, a certainty of purpose, and this can invite conditioned submission. ... Property—income provides the wherewithal to purchase submission.' (p. 6)

Galbraith continues to describe the character of wealth:

> 'In past time, so great was the prestige of property that ... it accorded power to its possessor. What the man of wealth said or believed attracted the belief of others as a matter course' (p.49). … To this day, 'wealth per se no longer gives automatic access to conditional power. The rich man who now seeks such influence hires a public relations firm to win others to his beliefs. Or he contributes to a political or a political action committee that reflects his views. Or he goes into politics himself and uses his property not to purchase votes but to persuade voters' (p. 50).

In ' *'The spirit capitalism' and long-run growth*', Heng-fu Zou (1994) points out the relationship between the important motivation in seeking power and wealth (or capital) accumulation:

> 'To define the utility function on both consumption and capital or wealth is also way to model man not only an economic animal, but also a political animal. Ever since Aristotle, we are taught that 'man is by nature an animal of intended to live in a polis'. Wealth or property provides man not only consumption means but also political power and social prestige. Procession of wealth is, to a considerable degree, a measure and standard of a person's success in a society.'

3.2. The model setup and multiply equilibria

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[6] Lord Acton, 1988, in: J.R. Fears, ed., Essays in religion, politics, and morality (Liberty classics, Indianapolis, IN) (p. 572)

In our modified model, wealth is included into the process of power accumulation. Formally, the model is given as

(3.1) $$\max\int_0^\infty U(c,p)\exp(-\rho t)dt$$

subject to

(3.2) $$\frac{da}{dt}=ra+Rp+1-T-c$$

(3.3) $$\frac{dp}{dt}=g(a,p,T)-\delta p$$

$$a(0)=a_0>0;\quad p(0)=p_0>0$$

where a is wealth, T is $T_2(t)$ as in section 2, which is the time and energy allocated to political activity, fight, compete, publicity-seeking; r is the revenue rate brought about by wealth; g is the power generation function, which now is a function of wealth, power, and the time and energy allocating to pursing power. Again, we assume g satisfy the following general properties:

$$g_1\geq 0\quad g_2\geq 0\quad g_3\geq 0\quad g_{11}\leq 0\quad g_{22}\leq 0\quad g_{33}\leq 0$$

The current-value Hamiltonian for the optimal problem (3.1), with 'prices' $\lambda_1(t)$ and $\lambda_2(t)$ used to value increments to wealth and power respectively, is

$$\mathrm{H}(a,p,\lambda_1,\lambda_2,c,T,t)=U(c,p)+\lambda_1[ra+Rp+1-T-c]+\lambda_2[g(a,p,T)-\delta p]$$

In this model, there are two decision variables—consumption, $c(t)$, and the time devoted to power, $T(t)$—and these are (in an optimal program) selected so as to maximize H. The first-order necessary conditions for this problem are thus

(3.4) $$U_c=\lambda_1$$

(3.5) $$\lambda_1=\lambda_2 g_3$$

on the margin, goods must be equally valuable in their two uses—consumption and wealth accumulation (eq. (3.4))—and time must be equally valuable in two uses—wealth and power accumulation (eq, (3.5)).

The rates of change of prices λ_1 and λ_2 are given by

(3.6) $$\dot{\lambda}_1=(\rho-r)\lambda_1-\lambda_2 f_1$$

(3.7) $$\dot{\lambda}_2=(\rho+\delta-g_2)\lambda_2-R\lambda_1-U_p$$

The usual two transversality conditions,

(3.8) $$\lim_{t\to\infty}e^{-\rho t}\lambda_1(t)a(t)=0$$

(3.9) $$\lim_{t\to\infty}e^{-\rho t}\lambda_2(t)p(t)=0$$

Then eqs. (3.2) and (3.3) and (3.4)-(3.7), together with two transversally conditions (3.8) and (3.9), implicitly describe the optimal evolution of $a(t)$ and $p(t)$ from any initial mix of these two kinds of variables.

The steady-state value of wealth, power, consumption, and time allocating to purse power, which are denoted by a^*,p^*,c^*, and T^* respectively, satisfy the following equations:

(3.10) $$(\rho-r)g_3(a^*,p^*,T^*)=g_1(a^*,p^*,T^*)$$

(3.11) $$\rho+\delta-g_2(a^*,p^*,T^*)=[R+\frac{U_p(c^*,p^*)}{U_c^*(c^*,p^*)}]g_3(a^*,p^*,T^*)$$

(3.12) $$ra^*+Rp^*+1-T^*-c^*=0$$

(3.13) $$g(a^*,p^*,T^*)=\delta\, p^*$$

From equations (3.10)-(3.13), it is widely known that this dynamic system could exist multiple equilibria in a very general case. To be intuitionistic, we give a general example that exists two equilibria.

Example: Assume $U(c,p)=vc^{\theta}+mp^{\vartheta}+np$

$$f(a,p,T)=\sigma p+\psi a^{\alpha}T^{\beta}$$

Let

$$v=3 \quad \theta=0.6 \quad m=0.2025 \quad \vartheta=0.79 \quad n=1.5 \quad \rho=0.65$$

$$\sigma=0.28 \quad \psi=0.6 \quad \alpha=0.65 \quad \beta=0.158 \quad R=0.25 \quad \delta=0.75 \quad r=0.36$$

By equations (3.10)-(3.13), we get two equilibria:

$$a_1^*=0.463808 \quad p_1^*=0.451295818 \quad T_1^*=0.032694892 \quad c_1^*=1.2471$$

$$a_2^*=8.59412 \quad p_2^*=4.774135682 \quad T_2^*=0.605819285 \quad c_2^*=4.6816$$

The existence of multiple equilibria imply that two people with quite close initial power and wealth endowment may consume, and allocate time between the pursuit of power and the production of wealth, at completely different growth path, they would converge to different steady states.

3.3. The existence of stable limit cycles

For this particular model, it's interesting to analyze the bifurcation of closed orbits emerged from the steady state. In this subsection we will investigate the conditions for the existence of closed orbits and stability. It is shown that if the parameter that measures the degree of desire for power crosses some values, stable limit cycles emerge around the stationary point.

In the following subsection, we keep the utility function separable as in Zou (1991, 1994), assuming that

$$U(c,s)=u(c)+\pi v(p)$$

where π, as in section 2, is a positive constant that measures the degree of desire for power.

We will apply the Hopf Bifurcation to establish limit cycles. This theorem considers the stability properties of a family of nonlinear dynamic systems for variations of a parameter. More precisely, this theorem states that the stable limits cycles exist if (i) two purely imaginary eigenvalues exist for a critical value of a parameter, such that (ii) the imaginary axis is crossed at nonzero velocity and that (iii) certain stability conditions are met. The appendix lists the precise requirements.

To prove that the existence of closed orbits arising from the particular model is generic and common, for simplicity, we assume a general and reasonable utility function:

(3.14) $$U(c,s)=\ln c+\pi\ln p$$

and a separable stress generation function:

(3.15) $$g(a,p,T) = Ap + BT^{\alpha} + Da$$

where *A, B* and *D* are all positive and $0 < \alpha < 1$.

The necessary conditions for these optimal control problems change to

(3.16) $$\frac{1}{c} = \lambda_1$$

(3.17) $$\lambda_1 = B\alpha T^{\alpha-1}\lambda_2$$

(3.18) $$\dot{\lambda}_1 = (\rho - r)\lambda_1 - D\lambda_2$$

(3.19) $$\dot{\lambda}_2 = (\rho - A + \delta)\lambda_2 - R\lambda_1 - \frac{\pi}{p}$$

(3.20) $$\lim_{t\to\infty} e^{-\rho t}\lambda_1(t)a(t) = 0$$

(3.21) $$\lim_{t\to\infty} e^{-\rho t}\lambda_2(t)p(t) = 0$$

together with dynamic equations (3.2) and (3.3).

Notice that the closed orbits as well as the paths that approach them are optimal in the sense that they comply with the 'transversality condition', since along the closed orbits *a*(*t*) and *p*(*t*) are bounded, (3.20) and (3.21) is verified for any $\rho > 0$.

From (3.16) and (3.17), we get two expressions of the two controls *c* and *T* in terms of two co-state variables λ_1 and λ_2. Replacing the *c* and *T* with their expressions into (3.2), (3.3), then together with (3.18) and (3.19), we get a four-dimensional dynamic system in *a*, *p*, λ_1 and λ_2:

(3.22a) $$\dot{a} = ra + Rp + 1 - \left(\frac{B\alpha\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha}} - \frac{1}{\lambda_1}$$

(3.22b) $$\dot{p} = (A - \delta)p + B\left(\frac{B\alpha\lambda_2}{\lambda_1}\right)^{\frac{\alpha}{1-\alpha}} + Da$$

(3.22c) $$\dot{\lambda}_1 = (\rho - r)\lambda_1 - D\lambda_2$$

(3.22d) $$\dot{\lambda}_2 = (\rho - A + \delta)\lambda_2 - R\lambda_1 - \frac{\pi}{p}$$

Then, the steady-state values of dynamic system (3.22), denoted by a^*, s^*, λ_1^* and λ_2^* satisfy:

(3.23a) $$ra^* + Rp^* + 1 - \left(\frac{B\alpha\lambda_2^*}{\lambda_1^*}\right)^{\frac{1}{1-\alpha}} = \frac{1}{\lambda_1^*},$$

(3.23b) $$(A - \delta)p^* + B\left(\frac{B\alpha\lambda_2^*}{\lambda_1^*}\right)^{\frac{\alpha}{1-\alpha}} + Da^* = 0,$$

(3.23c) $$(\rho - r)\lambda_1^* = D\lambda_2^*,$$

(3.23d) $$(\rho - A + \delta)\lambda_2^* = R\lambda_1^* + \frac{\pi}{p^*}.$$

We assume that the steady state exists. Linearization of dynamic system (3.23) around the steady state yields the following system:

$$\dot{z} = J(z^*)(z - z^*)$$

where $z = (a, p, \lambda_1, \lambda_2)^T \in \Re^4$, z^* is the steady state and $J(z^*)$ is the Jocobian evaluated at the steady state that is given by

$$(3.24) \qquad J(z^*) = \begin{pmatrix} r & R & \frac{1}{\lambda_1^{*2}} + \frac{1}{(1-\alpha)\lambda_1^*}\left(\frac{\lambda_2^*}{\lambda_1^*}\right)^{\frac{1}{1-\alpha}} & -\frac{1}{(1-\alpha)\lambda_1^*}\left(\frac{\lambda_2^*}{\lambda_1^*}\right)^{\frac{\alpha}{1-\alpha}} \\ D & A-\delta & -\frac{1}{(1-\alpha)\lambda_1^*}\left(\frac{\lambda_2^*}{\lambda_1^*}\right)^{\frac{\alpha}{1-\alpha}} & \frac{1}{(1-\alpha)\lambda_1^*}\left(\frac{\lambda_2^*}{\lambda_1^*}\right)^{\frac{2\alpha-1}{1-\alpha}} \\ 0 & 0 & \rho - r & -D \\ 0 & \pi/p^{*2} & -R & \rho - A + \delta \end{pmatrix}.$$

Conveniently, to derive $J(z^*)$, we have assumed $B\alpha = 1$ without effecting the main results.

It remains for us to show that for some values of the relevant parameters, the differential equation system that is given above has exactly two imaginary roots. In order to discuss this case, a few preliminary considerations are in point. Although bifurcation of a system can be studied in relation to a multi-dimensional parameter, and in a series of papers studying limit cycles, such as Benhabib (1978), Benhabib and Nishimura (1979), Dockner and Feichtinger (1991, 1993), Medio (1987), and Foley (1992) are all take the time discount rate (i.e. ρ in this paper) as the parameter, we shall only take the parameter as the degree of desire for power, π.

At this point we want to determine the conditions for $J(z^*)$ to exist a pair of pure imaginary eigenvalues, to do this, we make use of the following theorem.

Theorem 1: *The necessary and sufficient conditions for all eigenvalues to be complex and two having zero real parts* are:

$$(3.25) \qquad \det J(z^*) > \left(\frac{K}{2}\right)^2$$

$$(3.26) \qquad \det J(z^*) - \left(\frac{K}{2}\right)^2 - \rho^2\left(\frac{K}{2}\right) = 0$$

where $K \equiv M_2 - \rho^2$ *and* M_2 *is the sum of the principal minors of second order of* $J(z^*)$.

Proof: See Dockner and Feichtinger (1991).

To apply Theorem 1, let us start with the determinant. The determinant of $J(z^*)$ can be explicitly computed as follows:

$$(3.27) \qquad \det J(z^*) = (\rho - r)\left(\rho + \delta - A - \frac{DR}{\rho - r}\right)\left[r(A - \delta) - DR\right] + \frac{D^2\pi}{p^{*2}\lambda_1^{*2}}$$

K can be written as

$$(3.28) \qquad K = r(\rho - r) - 2DR - (\delta - A)(\rho + \delta - A) - \frac{\pi}{(1-\alpha)p^{*2}\lambda_1^*}\left(\frac{\rho - r}{D}\right)^{\frac{2\alpha-1}{1-\alpha}}$$

Since we take the degree of abhorrence to stress, π, as the only parameter, det $J(z^*)$ and K are only the functions of π. Imposing conditions (3.25) and (3.26), we get the critical value of $\hat{\pi}$, from which a family of closed orbits emerge by Hopf bifurcation Theorem. We have the following Proposition:

Proposition 1: *Let the optimal control problem* (3.1) *satisfy assumption* (3.14)-(3.15). *For* $\pi = \hat{\pi}$ *let the Jacobian* $J(z^*)$ *have one pair of imaginary roots* $\xi(\hat{\pi}) \pm \mu(\hat{\pi})i$, *where* $\xi(\hat{\pi}) = 0$, $\mu(\hat{\pi}) \neq 0$, $d\xi(\hat{\pi})/d\pi \neq 0$. *Then for the optimal problem given by* (3.1) *there exist a continuous function* $\pi = \pi(\varepsilon)$, $\pi(0) = \hat{\pi}$, *and a* C^∞ *family of optimal paths* $(c(t,\pi(\varepsilon)),\ T(t,\pi(\varepsilon)),\ a(t,\pi(\varepsilon)),\ p(t,\pi(\varepsilon)))$ *that are nonconstant closed orbits in the positive orthant for sufficiently small* $\varepsilon \neq 0$.

Proof: Setting $\pi = \hat{\pi}$ in (3.23), by implicit function, we can yield steady-state values of c, T, a, and p as C^∞ function of $\hat{\pi}$. The Hopf Bifurcation Theorem immediately applies and we obtain closed orbits.

Under our assumptions the steady state is interior, that is, steady-state values of $c(\pi)$, $T(\pi)$, $a(\pi)$, and $p(\pi)$ are positive. By the Hopf Bifurcation Theorem, the orbits collapse into the stationary point as $\varepsilon \to 0$ and $\pi(\varepsilon) \to \hat{\pi}$. Then for sufficiently small ε the orbits $(c(t,\pi(\varepsilon)),\ T(t,\pi(\varepsilon)),\ a(t,\pi(\varepsilon)),\ p(t,\pi(\varepsilon)))$ remain in the positive orthant.

Finally, the optimality of a path that forms or approaches the orbit is assured by the transversality conditions (3.20)-(3.21), which as we have revealed are satisfied.

Q.E.D

Thus we have revealed that if we choose appropriate parameters, a family of closed orbits, which is capable of explaining cyclical consumption, power, competing and fighting, and asset expressed as limit cycles, will emerge from the dynamic system. Now we present a numerical example that establishes a stable limit cycles. The limit cycles can be identified persistent oscillatory behavior and hence is capable of explaining the continuous changing of power throughout much of a person's lifetime.

For the numerical example we make uses of the following parameter values:

$$\begin{array}{llll} \alpha = 0.75 & B = 4/3 & \rho = 0.68 & r = 0.4 \\ D = 0.4 & A = 0.54 & \delta = 0.56 & R = 0.11672 \end{array} \tag{3.29}$$

Let $\pi = 2.69618$, then the roots of the $J(z^*)$ will be $0 \pm 0.014935\, i$, $0.68 \pm 0.014935\, i$, and $\left.\dfrac{d\xi(\pi)}{d\pi}\right|_{\pi=2.69618} = 0.100946$. The steady-state quantities at $\pi = 2.69618$ are given below:

$$p^* = 175.124 \quad T^* = 0.2401 \quad a^* = 7.61267 \quad c^* = 24.245 \tag{3.30}$$

Proposition 2: *If the parameters are specified as in* (3.29) *and the degree of desire for power assumes the critical value* $\hat{\pi} = 2.69618$, *then there exists a pair of imaginary roots that gives rise to the local existence of limit cycles. These cycles are stable and occur for an interval of* π *slightly greater than* $\hat{\pi}$.

Proof: The existence of limit cycles is a straight inference of Proposition 1. The proof of the stability of these cycles is tedious, a detail outline is put on appendix.

Q.E.D

Proposition 2 shows the existence of stable limit cycles as rational intertemporal consumption choices. The closed orbits are depicted in Figures 5. These cycles are generic in the sense that they do occur for an interval of the parameter values of desire for power (see Guckenheimer and Homes, 1990). Hence, Limit cycles are not knife-edge, as perhaps suggested by critical val.

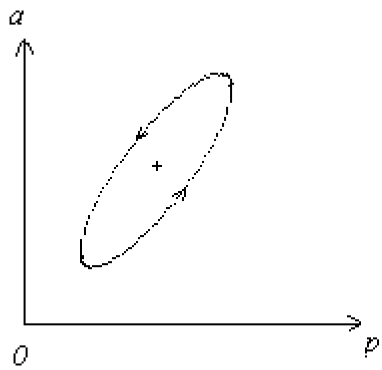

Figure 5. Cycle pattern in the state plane (*a, p*)

4. Conclusion

In this paper, we have offered two positive intertemporal general equilibrium models that sheds considerable light on the dynamic property of power. Our results consist of the following: people with the same preference and the same time discount rate may have different steady state and different growth path depending on the difference in their power and wealth endowment. The cyclical power patterns can be explained by the rational behavior of pursuing power in a "perfect" world. Numerical simulation provided strong support for these modelings.

Appendix: Proof of Stability of Limit Cycles

To prove the stability of limit cycles claimed at proposition 2, we restate the following theorem (cf. Guckenheimer and Holmes, 1990)

Hopf Bifurcation Theorem: *Suppose that the system* $\dot{x} = f_{\mu}(x),\ x \in \Re^n,\ \ \mu \in \Re$ *has an equilibrium* (x_0, μ_0) *at which the following properties are satisfied:*

(*H1*) $D_x f_{\mu_0}(x_0)$ *has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero parts.*

Then (H1) implies that there is a smooth curve of equilibria $(x(\mu), \mu)$ *with* Re $a_{2,1}$. *The eigenvalues* $\lambda(\mu), \overline{\lambda}(\mu)$ *of* $D_x f_{\mu_0}(x(\mu))$ *which are imaginary at* $\mu = \mu_0$ *vary smoothly with* μ. *If, moreover,*

(*H2*) $$\frac{d}{d\mu}(\mathrm{Re}\,\lambda(\mu))\Big|_{\mu=\mu_0} = d \neq 0$$

then there is a unique three-dimensional center manifold passing through (x_0, μ_0) *in* $\Re^n \times \Re$ *and a smooth system of coordinates (preserving the planes* μ *= const.) for which the Taylor expansion of degree 3 on the center manifold is given by the following normal form:*

(A.1) $$\dot{x} = (d\mu + e(x^2 + y^2))x - (\omega + c\mu + b(x^2 + y^2))y,$$

(A.2) $$\dot{y} = (\omega + c\mu + b(x^2 + y^2))x + (d\mu + e(x^2 + y^2))y,$$

or is expressed in polar coordinates as

$$\dot{r} = (d\mu + er^2)r,$$

$$\dot{\theta} = (\omega + c\mu + br^2).$$

If $e \neq 0$, *there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of* $\lambda(\mu_0), \overline{\lambda}(\mu_0)$ *agreeing to second order with the paraboloid* $\mu = -(e/d)(x^2 + y^2)$. *If* $e < 0$, *then these periodic solutions are stable limit cycles, while if* $e > 0$, *the periodic solutions are repelling.*

To derive an analytical expression for the stability parameter *e* of the normal form (A.1) and (A.2) we will apply several transformations to the canonical system (3.22) in the following way: First, we changes the coordinates according to

(A.3) $$\zeta_1 = a - a^*, \quad \zeta_2 = p - p^*, \quad \zeta_3 = \lambda_1 - \lambda_1^*, \quad \zeta_4 = \lambda_2 - \lambda_2^*$$

This procedure simply transforms the equilibrium to the origin. The canonical differential equation system (3.22) is now given in the new coordinates ζ by

(A.4) $$\dot{\zeta} = J\zeta + \phi(\zeta)$$

where *J* is the Jacobian (3.24) and $\phi(\zeta)$ is given by

(A.5) $$\phi(\zeta)=\begin{bmatrix} -15663\zeta_3^{\ 2}+3225.95\zeta_3\zeta_4-1728.19\zeta_4^2 \\ 1612.9\xi_3^{\ 2}-3456.37\zeta_3\zeta_4+1645.89\zeta_4^2 \\ 0 \\ -5.02044\times10^{-7}\zeta_2^2 \end{bmatrix}$$

The Jacobian possesses the eigenvalues, $0.68+0.014935\,i$, $0.68-0.014935\,i$, $-0.014935\,i$ and $0.014935\,i$, with the corresponding eigenvectors $E:=(e_1,e_2,e_3,e_4)$. Using the eigenvectors as a basis for a new coordinate system, we set

(A.6) $$\upsilon := E\zeta$$

According to this transformation the time derivative of new coordinates x are computed from

(A.7) $$\dot{\upsilon}=E^{-1}JE\upsilon+E^{-1}\phi(E\upsilon)$$

with

$$E^{-1}JE=diag(0.68+0.014935i,\,0.68-0.014935i,\,0.014935i,\,0.014935i).$$

In order to reduce the four-dimensiponal system to the central manifold (where only the crtical variable and its conjugate matter, i.e., two-dimensional) and to transform the differential equation for the critical variable υ_3 to its canonical form:

(A.8) $$\dot{\upsilon}_3=0.014935i\upsilon_3+\sum_{j=1}^{\infty}e_{j+1,j}\upsilon_3\left|\upsilon_3\right|^{2j}$$

we may write the time derives of υ_3 and υ_2 as

(A.9) $$\dot{\upsilon}_3=0.014935i\upsilon_3+\frac{1}{2}g_{\upsilon_3\upsilon_3}\upsilon_3^2+g_{\upsilon_3\upsilon_4}\upsilon_3\upsilon_4+\frac{1}{2}g_{\upsilon_4\upsilon_4}\upsilon_4^2+g_{\upsilon_3\upsilon_1}\upsilon_3\upsilon_1$$
$$+g_{\upsilon_3\upsilon_2}\upsilon_3\upsilon_2+g_{\upsilon_4\upsilon_1}\upsilon_4\upsilon_1+g_{\upsilon_4\upsilon_2}\upsilon_4\upsilon_2$$

(A.10) $$\dot{\upsilon}_2=(0.68-0.014935i)\upsilon_2+\frac{1}{2}g_1\upsilon_3^2+g_2\upsilon_3\upsilon_4+\frac{1}{2}g_4\upsilon_4^2$$

Note that (A.8) up to third-order terms (i.e., j=1) is equivalent to the normal form (A.3), (A.4) by setting $\upsilon_3=x+iy$. The stability parameter e of (A.3) and (A.4) equals $\mathrm{Re}\,e_{2,1}$ of (A.8). Therefore our main goal is to derive an analytical expression for $\mathrm{Re}\,e_{2,1}$ for our model to determine the stability of the cycles.

Comparing the coefficients of (A.7) with (A.8) yields

(A.11)
$$g_{\upsilon_1\upsilon_3}=-22.0017+345.157i,$$
$$g_{\upsilon_2\upsilon_3}=3159.19+4989.81\,i,$$
$$g_{\upsilon_3\upsilon_3}=-0.624172-1.225886\,i,$$
$$g_{\upsilon_1\upsilon_4}=-1.96472+248.677\,i,$$
$$g_{\upsilon_2\upsilon_4}=2397.29+3378.7\,i,$$
$$g_{\upsilon_3\upsilon_4}=-0.482948-0.839527\,i,$$
$$g_{\upsilon_4\upsilon_4}=-0.36945-0.5771484\,i,$$
$$g_1=0.000465602-0.0035956i,$$
$$g_2=0.000191262-0.00254312i,$$
$$g_3=0.000038961-0.001791616i.$$

The next step is to reduce the system to the central manifold. The uncritical variable υ_2 on the central manifold is given by the critical variable υ_3 by a quadratic (or more precisely, approximation since the higher-order terms do not matter):

(A.12) $$\upsilon_2 = h(\upsilon_3, \upsilon_4) = \frac{1}{2}h_1\upsilon_3^2 + h_2\upsilon_3\upsilon_4 + \frac{1}{2}h_3\upsilon_4^2 + \cdots$$

For the flow on the central manifold the relation

(A.13) $$\dot{\upsilon}_2 = (\partial h / \partial \upsilon_3)\dot{\upsilon}_3 + (\partial h / \partial \upsilon_4)\dot{\upsilon}_4$$

must hold. Comparing (A.13) and (A.10) yield

(A.14) $$\begin{aligned} h_1 &= -0.00102806 + 0.00521103i \\ h_2 &= -0.000363232 + 0.0037319i \\ h_3 &= 0.00263472i \end{aligned}$$

Substituting $\upsilon_2 = h(\upsilon_3, \upsilon_4)$ into (A.9), we obtain

(A.15) $$\begin{aligned} \dot{\upsilon}_3 = 0.014935i\upsilon_3 &+ \frac{1}{2}g_{\upsilon_3\upsilon_3}\upsilon_3^2 + g_{\upsilon_3\upsilon_4}\upsilon_3\upsilon_4 + \frac{1}{2}g_{\upsilon_4\upsilon_4}\upsilon_4^2 + g_{\upsilon_3\upsilon_1}\upsilon_3\overline{h}(\upsilon_3, \upsilon_4) \\ &+ g_{\upsilon_3\upsilon_2}\upsilon_3 h(\upsilon_3, \upsilon_4) + g_{\upsilon_4\upsilon_1}\upsilon_4\overline{h}(\upsilon_3, \upsilon_4) + g_{\upsilon_4\upsilon_2}\upsilon_4 h(\upsilon_3, \upsilon_4) \end{aligned}$$

where the bar means complex conjugate.

Following Guckenheimer and Holmes (1983) the contribution of the quadratic terms to $\mathrm{Re}\, a_{2,1}$ is

(A.16) $$\mathrm{Re}\{ig_{\upsilon_3\upsilon_3}g_{\upsilon_3\upsilon_4} / (2 \times 0.014935)\}$$

Summing up the contributions of the remaining terms of (A.15) to $\mathrm{Re}\, e_{2,1}$ and (A.16), we compute

(A.17) $$\mathrm{Re}\, e_{2,1} = \mathrm{Re}\{33.478406ig_{\upsilon_3\upsilon_3}g_{\upsilon_3\upsilon_4} + g_{\upsilon_3\upsilon_1}\overline{h}_2 + \frac{1}{2}g_{\upsilon_1\upsilon 4}\overline{h}_1 + g_{\upsilon_2\upsilon_3}h_2 + \frac{1}{2}g_{\upsilon_2\upsilon_4}h_1\}$$

And after substituting (A.11) and (A.14),

(A.18) $$\mathrm{Re}\, e_{2,1} = -65.2233$$

Thus, the existence of stable limit cycles generated by a Hopf bifurcation is guaranteed as long as $\hat{\pi} = 2.69618$ and provided that π is chosen slightly larger than $\hat{\pi}$ (Note that the real part of the roots increases with π: $d\xi(\pi)/d\pi > 0$, Thus as π passes through the value 2.69618, the steady state changes from saddle-point stable to totally unstable). This completes the proof of proposition 2.

Q.E.D.

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