Multiple Equilibria and Indeterminacy in an Optimal Growth Model with Endogenous Capital Depreciation

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Abstract

This paper extends an otherwise standard one-sector neoclassical growth model by postulating that the depreciation rate of physical capital depends on the agent's efforts on maintenance and repairs. Specifically, we introduce endogenous depreciation into the standard optimal growth model via two different mechanisms and examine the steady state and the dynamics of the model economy qualitatively and quantitatively. We find that with plausibel parameter values, multiple equilibria and indeterminacy can arise in simply modified optimal growth model.

Keywords: Multiple Equilibria, Indeterminacy, Endogenous Depreciation

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1 Introduction

Dynamic general equilibrium theory has investigated the possibility of multiplicity of the equilibria and indeterminacy of equilibrium paths. Multiplicity of the equilibria and indeterminacy can explain why fundamentally similar economies exhibit the same per capita income but different growth rates, or why economies with the same growth rate can exhibit different per capita levels of income. This paper explores the implication on the economic equilibria of the assumption that the agent chooses her depreciation rates endogenously by spending resources and time on maintaining the stock of the physical capital. We study the complex dynamic behavior of the neoclassical growth model with endogenous depreciation.

The standard neoclassical growth models, either the Solow model or the Ramsey model, simply assumes that the depreciation rate is an exogenously positive constant. And the model economy shows nice convergency properties of the unique steady state. As a matter of fact, their models implicitly assume that the agent can't change the depreciation rate through maintenance and repairs, and the maintenance expenditure is not an independent variable in their models. In other words, maintenance and repairs do not matter for the accumulation of the physical capital. However, it is truth that the machine may be used longer if the agent spends some time and resources to maintain and repair them constantly and regularly. That is to say, the depreciation rate may be reduced endogenously through maintenance and repairs. And this paper wants to formulate this idea and examines how the results of the standard models will be changed.

In the literature, many papers have considered the endogeneity of depreciation rates and optimal maintenance of the physical capital. Empirically, Bitros (1976) obtains empirical results that maintenance expenditures are significantly related to gross investment and scrappage as well as other cyclically sensitive variables and their trade-offs are substantial. Therefore, he advises that maintenance expenditures be included among the independent variables in the econometric models about investment. Theoretically, Auernheimer (1986) examines the robustness of the conventional results concerning the relationship between the interest rate and the price of capital, and the relationship among total capital services, employment and output with variable depreciation rates. In his paper, the rate of depreciation of the capital stock is the increasing and concave function of the intensity of use. Recently, Rioja (2003) examines the maintenance of existing public infrastructure in developing countries, which endogenizes the depreciation rate of the existing public infrastructure. The quantitative results of his paper show that reallocating funds from new infrastructures to maintenance can have positive effects on those countries' GDPs. And, by introducing optimal maintenance and a linear depreciation function, Dangl and Wirl (2004) shows how to solve the Bellman equation analytically. In a paper, Gylfason & Zoega (2007) defines the depreciation rate as a decreasing function of the durability of the capital stock and explain the differences in the quality of physical capital across countries. And with the endogeneity of capital depreciation, Mukoyama (2008) finds that the acceleration of investment-specific technological progress distorts the measurement of the aggregate capital stock and accounts for a large portion of the observed productivity slowdown since the 1970s. Furthermore, by assuming the depreciation rate as a strictly increasing function of the rate of capital utilization and a strictly decreasing function of maintenance expenditure, Fujisaki & Mino (2009) examines the long-run effects of inflation tax in a cash-in-advance economy. However, the mechanism of endogeneity of these papers is different from ours. The basic law of our endogenous mechanism is that the current depreciation rate is a deceasing and convex function of the maintenance cost of the physical capital.

The literature on multiple equilibria and indeterminacy is large. Kurz (1968) puts forward the possibility of multiple equilibria in an optimal growth model with wealth effects. The channel that he reaches the multiplicity of the equilibria is putting the state variable (physical capital) into the utility function. And, Boldrin and Montrucchio (1986) prove the indeterminacy of the optimal capital accumulation paths for the small enough discount parameters. In a model of industrialization, Matsuyama (1991) shows that multiple steady states exist because of the increasing returns in the manufacturing sector. In another research, Evans, Honkapohja & Domer (1998) constructs a rational expectation model, in which monopolistic competition and complementarities between types of capital goods induce the expectational indeterminacy. The models of many papers display indeterminate steady states because of aggregate increasing returns generated by externalities or monopolistic competition or both, such as Murphy, Schleifer and Vishny (1989), Spear (1991), Howitt and McAfee (1992), Kehoe, Levine and Romer (1992), Benhabib and Farmers (1994), and many other researches. Benhabib, Meng and Nishimura (2000) obtains indeterminacy under constant returns to scale in multisector economies.

The rest of this paper is organized as follows. Section 2 describes the neoclassical growth model with a simple mechanism of endogenous depreciation, examines its complex dynamic behavior, and solves for the steady states numerically. Section 3 discusses a different endogenous mechanism of depreciation, studies the characteristics of a four-dimensional dynamic system and presents the numerical solutions. Finally, the concluding remarks are presented in section 4.

2 The Neoclassical Growth Model with Endogenous Depreciation

2.1 The Simple Model with Endogenous Depreciation

We consider a macroeconomic model with identical infinitely lived representative agent. The representative agent chooses her consumption path c_t, capital accumulation path k_t and the resources spending path on maintaining the machinery or physical capital s_t, to maximize the discounted utility, namely,

$$\max_{\{c,s,k\}} \int_0^\infty u(c)\, e^{-\rho t} dt, \tag{1}$$

and subjects to the initial positive capital stock $k(0)$ and the budget constraint (for simplicity, suppose that the population growth rate is zero.)

$$\dot{k} = f(k) - c - s - \delta(s)\, k, \tag{2}$$

where ρ is the positive time preference rate. $u(c)$ is the instantaneous utility function defined on the private consumption $c(t)$, and the utility function is strictly increasing, strictly concave, namely $u_c > 0$, $u_{cc} < 0$. And $f(k)$ is the standard neoclassical production function, satisfying the following neoclassical properties, $f'(k) > 0$, $f''(k) < 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f''(k) = 0$.

Furthermore, $\delta(s)$ is the depreciation rate, which is endogenously determined by the agent's spending on maintaining the physical capital s. Following Fujisaki & Mino (2009) with simplification, we assume that $\delta(s)$ is a decreasing and convex function of s: $\delta'(s) < 0$, $\delta''(s) > 0$, $\forall s$. That is to say, the more the resources being spent on maintenance, the less the depreciation rate; but the rate of decrease of the depreciation rate is decreasing. As a matter of fact, we suppose implicitly the depreciation rate of period t depend upon the expenditure on maintenance of period t only and the depreciation function is time invariant. As always, $\delta \in [0,1]$ in a general way.

The Hamiltonian for the above optimization problem can be written as follows:

$$H = u(c) + \lambda\left[f(k) - c - s - \delta(s)\, k\right],$$

The first-order conditions for optimization are:

$$u'(c) = \lambda, \tag{3}$$

$$-1 - \delta'(s)\, k = 0, \tag{4}$$

$$\dot{\lambda} = -\left[f'(k) - \delta(s) - \rho\right]\lambda, \tag{5}$$

and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t}\lambda k = 0,$$

where λ is the co-state variable for physical capital k. Equation (3) is the familiar optimal condition, which states that the marginal utility of consumption is equal to the marginal value of capital stock. Equation (4) gives the optimal maintenance cost which is an increasing function of the current capital stock. And equation (5) is the familiar Euler equation which determines the intertemporal choice of consumption and maintainance.

2.2 Dynamic System

In this subsection, we present the dynamics of the system. From equation (3) and (5), we obtain

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - \delta(s) - \rho\right]. \tag{6}$$

Totally differentiating equation (4) yield

$$\frac{ds}{dk} = -\frac{\delta'(s)}{k\delta''(s)}, \tag{7}$$

which is positive because of the properties of the depreciation function. Hence, s is an increasing function of k, i.e., $s = s(k)$ and $s'(k) > 0$. However, we cannot obtain the sign of the second derivative of this function because $s''(k)$ relies on the third derivative of the depreciation function. In fact, this is the reason why we can obtain very complex dynamics in this model, about which we will talk in the subsequent sections.

Substituting $s = s(k)$ into equation (6) and (2), we obtain the following dynamic system in the (c, k) space and this system completely characterizes our model economy:

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - \delta(s(k)) - \rho\right], \tag{8}$$

$$\dot{k} = f(k) - c - s(k) - \delta(s(k))\, k. \tag{9}$$

2.3 Steady State

Let $\dot{c} = \dot{k} = 0$. The steady-state levels of the economy, (c^*, k^*) can be characterized by

$$f'(k^*) = \delta(s(k^*)) + \rho, \tag{10}$$

$$f(k^*) = c^* + s(k^*) + \delta(s(k^*))\, k^*. \tag{11}$$

At first, we examine the existence of the equilibria determined by equation (10). Then we study the properties of this algebraic equation. Define $\phi(k) = f'(k)$, $\psi(k) = \delta(s(k)) + \rho$. Then the neoclassical production function gives $f'(k) > 0$, $f''(k) < 0$ and Inada conditions

$$\lim_{k\to\infty} f'(k) = 0, \lim_{k\to 0} f'(k) = \infty.$$

Then, we can have the following properties of $\phi(k)$:

$$\lim_{k\to 0} \phi(k) = \lim_{k\to 0} f'(k) = \infty,$$
$$\lim_{k\to\infty} \phi(k) = \lim_{k\to\infty} f'(k) = 0,$$
$$\phi'(k) = f''(k) < 0.$$

However, we cannot obtain the exact sign of the second derivative of $\phi(k)$, because it is determined by the third derivative of the production function. Then in the coordinate space of $(k, \phi(k))$, we get a declining curve without any knowledge about its curvature. Meanwhile, from the bounded, decreasing and convex depreciation function and (7), we have the following properties of $\psi(k)$:

$$\psi'(k) = \delta'(s(k))\, s'(k) < 0, \psi(k) \in [\rho, 1+\rho].$$

Similarly, we do not know the exact sign of $\psi''(k)$ without any assumption about the third derivative of the depreciation function. Hence, we just get a declining curve in the coordinate space of $(k, \psi(k))$ without any information about its curvature.

Based on the aforementioned discussions, we cannot obtain the exact results about the existence of the steady state. But, we can conjecture all sorts of possibilities: one steady state, multiple steady states or a continuum of steady state. And we will present numerical solutions of these possibilities in section 2.5.

2.4 Stability of the Economic System

It is hard to get the explicit result about the global stability of the dynamic system with the initial capital stock and the transversality condition. However, we can draw conclusions on the local stability of the steady states. Based on the analysis of section (2.3) and the subsequent sections about numerical solutions, we know that the steady state may not exist. Naturally, we assume that there exists at least one steady state in this subsection. Then, linearizing system around the steady state (c^*, k^*), leads to

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \begin{pmatrix} 0 & -u'(c^*)\left[f''(k^*) - \delta'(s(k^*))\, s'(k^*)\right]/u''(c^*) \\ -1 & f'(k^*) - s'(k^*) - \delta'(s(k^*))\, s'(k^*)\, k^* - \delta(s(k^*)) \end{pmatrix} \begin{pmatrix} c - c^* \\ k - k^* \end{pmatrix}. \quad (12)$$

Denote J the coefficient matrix of the above system. The determinant and trace of the Jacobian matrix are

$$\det(J) = -u'(c^*)\left[f''(k^*) - \delta'(s(k^*))\, s'(k^*)\right]/u''(c^*),$$

and

$$\operatorname{trace}(J) = f'(k^*) - s'(k^*) - \delta'(s(k^*))\, s'(k^*)\, k^* - \delta(s(k^*)),$$

respectively.

To derive the stability of the system, the characteristic equation of the system is

$$\theta^2 + B\theta + C = 0,$$

where

$$B = -(\theta_1 + \theta_2) = -\operatorname{trace}(J), C = \theta_1\theta_2 = \det(J).$$

The characteristic roots of the system are

$$\theta_{1,2} = \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4C}\right).$$

Note that there is only one predetermined variable, k, in the system. Consequently, if the Jacobian matrix has two eigenvalues which have negative real parts, the linearized system is locally indeterminate. And if there is exactly one eigenvalue with negative real part, the system is saddle-point stable. Finally, if there is no eigenvalue with negative real part, the system is unstable. So we obtain the proposition.

Proposition Assume the steady states exist.

1), If $B^2 - 4C > 0$ and $\det(J) = \theta_1\theta_2 < 0$, then one eigenvalue is positive, and the other one is negative. Hence, the system is saddle-point stable.

2), If $B^2 - 4C \geq 0$, $\operatorname{trace}(J) = -(\theta_1 + \theta_2) > 0$, and $\det(J) = \theta_1\theta_2 < 0$, two eigenvalues have positive real roots; or if $B^2 - 4C < 0$ and $\operatorname{trace}(J) = -(\theta_1 + \theta_2) > 0$, two eigenvalues are conjugate complex roots with positive real parts. Hence, the system is totally unstable.

3), if $B^2 - 4C \geq 0$, $\operatorname{trace}(J) = -(\theta_1 + \theta_2) > 0$, $\det(J) = \theta_1\theta_2 > 0$ hold, two eigenvalues are negative real numbers; if $B^2 - 4C < 0$ and $\operatorname{trace}(J) = -(\theta_1 + \theta_2) > 0$, two eigenvalues are conjugate complex roots with negative real parts. Hence, the system is stable. Furthermore, the steady state is indeterminate, and in fact, we have a continuum of equilibria.

4), If $B^2 - 4C < 0$ and $\operatorname{trace}(J) = -(\theta_1 + \theta_2) = 0$ hold, two eigenvalues are conjugate complex roots with zero real part, then the system shows oscillating dynamics. In this case, the system neither converges nor diverges, and the trajectories are ellipses around the steady state.

2.5 Numerical Solutions

2.5.1 Example 1: Uniqueness of the Equilibrium

Case 1: The Saddle-point Stability of a Unique Equilibrium We take the utility function, the production function, the depreciation function, and the time preference rate as follows:

$$u(c) = \log c, \delta(s) = \frac{1}{1+s}, f(k) = 2.5k^{0.5}, \rho = 0.02. \tag{13}$$

After calculations, we have $s(k) = k^{0.5} - 1$, $s'(k) = 0.5k^{-0.5}$, $\delta(s(k)) = k^{-0.5}$. Then, we can find that there exists a unique equilibrium:

$$k^* = 156.25, c^* = 7.25.$$

And the associated optimal maintenance expenditure and optimal depreciation rate are:

$$s^* = 11.5, \delta^* = 0.08.$$

The eigenvalues of the Jacobian matrix in the case are -0.0554 and 0.0754. Therefore, the equilibrium is saddle-point stable. (Insert Figure 1.1 about here)

Case 2: Instability of a Unique Equilibrium If taking those functions as follows:

$$f(k) = 5k^{0.5}, u(c) = \log c, \delta(s) = \frac{1}{1+2s}, \rho = 0.05, \tag{14}$$

then, we obtain, $k^* = 1285.7864$, $c^* = 129.0786$, $s^* = 7.5336$, and $\delta^* = 0.0622$. The corresponding eigenvalues of the Jacobian are $0.0300 \pm 557.84i$. Thus, the steady state is unstable. (Insert Figure 1.2 about here)

2.5.2 Example 2: Multiple Equilibria and Local Indeterminacy

Case 1: Two Steady States and Local Indeterminacy The first one takes parameter values: $A = 5$, $\alpha = 0.3$, $\rho = 0.08$, and with the same functional forms as (14). We can get two steady states. One is $k_1^* = 229.7970$, $c_1^* = 4.6108$, $s_1^* = 10.2191$, and $\delta_1^* = 0.0466$. And the corresponding eigenvalues are $-0.0066 \pm 43.4891i$, which implies that we have two eigenvalues with negative parts. Hence, we conclude that this steady state is locally indeterminate. That is to say, there exists a continuum equilibrium locally. The other is $k_2^* = 9.5761$, $c_2^* = 5.9712$, $s_2^* = 1.6882$, and $\delta_2^* = 0.2285$. And the corresponding eigenvalues are $0.0400 \pm 10.1063i$. Therefore, the steady state is unstable. (Insert Figure 2.3 about here)

The second one takes parameter values: $A = 4$, $\alpha = 0.36$, $\rho = 2$, and with the same functional forms as (14). Then, one steady state is $k_1^* = 1.2670$, $c_1^* = 3.1746$, $s_1^* = 0.2507$, and $\delta_1^* = 0.6661$. The corresponding eigenvalues are: $0.3339 \pm 2.4968i$. The other is $k_2^* = 0.2641$, $c_2^* = 2.2506$, $s_2^* = -0.1366$, and $c_2^* = 2.2506$. And the corresponding eigenvalues are: $1.0000 \pm 2.3996i$. Therefore, both of these two equilibria are locally unstable. (Insert Figure 2.4 about here)

Case 2: Four Steady States and Local Indeterminacy The first example takes parameter values $A = 2$, $\alpha = 0.3$, $\rho = 0.05$, and with the same functional form as (14). We obtain four steady state $(k^*, c^*, s^*, \delta^*)$:

$$(321.2989, 3.4021, 12.1748, 0.0394), \quad (84.7063, 6.4226, 0.0079, 0.0768),$$
$$(0.5802, 3.6694, 0.0386, 0.9283), \quad (0.3576, 3.3270, -0.0772, 1.1825).$$

The eigenvalues of the Jacobian matrix associated with the above four equilibria are,

$$-0.0144 \pm 27.9367i, -0.0250 \pm 19.7090i, -0.0250 \pm 2.1107i, 0.0250 \pm 2.5198i.$$

Therefore, we can draw the conclusion that the former three steady states are locally indeterminate, whereas the fourth is unstable.

The second one takes the parameter value $A = 3$, $\alpha = 0.35$, $\rho = 0.02$, and with the same functional form with (13). We obtain four steady states:

$$(1.6464, 2.0057, 0.2831, 0.7794), \quad (1.1983, 2.0068, 0.0947, 0.9135),$$
$$(980.8246, -28.2030, 30.3181, 0.0319), \quad (4225.8150, -73.2696, 64.0063, 0.0154).$$

The eigenvalues of the Jacobian matrix associated with above four equilibria are,

$$-1.0474, 1.0274; -1.3246, 1.3446; -0.0100 \pm 0.0241i; -0.0054 \pm 0.0049i.$$

Hence, we know that the former two equilibria are saddle-point stability and the later two equilibria are locally indeterminate.

2.5.3 Example 3: One Steady State and Neutral Oscillating Dynamics

Let

$$u(c) = \log c, f(k) = 2k^{0.35}, \delta(s) = e^{-s}, \text{ and } \rho = 0. \tag{15}$$

The corresponding steady state is

$$(k^*, c^*, s^*, \delta^*) = (2.7706, 0.8380, 1.0191, 0.3609),$$

and the eigenvalues of the Jacobian matrix are $\pm 0.1955i$.

We know that if we assume zero time preference rate and nonzero depreciation rate or population growth rate in the standard neoclassical growth model, we can obtain only one steady state (excluding the degenerate zero steady state) and the system may converge or diverge to the steady state because the transversality condition may not help us find the optimal consumption at this time. But we can't obtain oscillating dynamics in the standard model. In our endogenous depreciation model, the oscillating dynamics emerge. At this time, the system neither converges nor diverges and the trajectories are ellipses around the steady state.

2.5.4 Example 4: No Steady State

Finally, we give an example in which there exists no steady state. If we assume $A = 0.5$, $\rho = 0.02$ and others are the same with (15), we can't get a real root in the real number space. That is to say, there does not exist a steady state.

3 Extension of the Benchmark Model

3.1 The Model with More General Mechanism of Endogeneity

In this section, we adopt a different endogenous mechanism of depreciation, certainly, more general. In the above simple model, we assume that the depreciation rate δ_t of time t depends on the maintenance cost s_t of time t only, i.e., $\delta_t = \delta(s_t)$. That is to say, there is no accumulation of the depreciation, or in other words, the current depreciation rate is independent of all of the past expenditures. But in reality, for example, a car with regular and aborative maintenance must be used much longer than the one with inattentive care. Therefore, we assume that the current depreciation rate depends on all of the past cares, mathematically,

$$\Delta_t = \int_{v=0}^{t} \tau(v,t)\,\delta(s_v)\,dv. \tag{16}$$

Δ_t is the depreciation rate of time t. s_v is the maintenance expenditure of time v. $\delta(s_v) = \delta_v$ is the depreciation function of time v, and it is a decreasing, concave function of s_v. We assume $\delta_v \in [0,1]$ as before. In order that the integral in (16) converges, we assume that $\tau(v,t) = ae^{b(v-t)}$, and $\int_{v=0}^{t} \tau(v,t)\,dv = 1$, where a, b are undetermined coefficients. After simple calculations, we can obtain

$$a \equiv \frac{b}{1-e^{-bt}}.$$

Furthermore, we have

$$\Delta_t = \int_{v=0}^{t} \frac{b}{1-e^{-bt}} e^{b(v-t)} \delta\left(s_v\right) dv.$$

Taking derivative about t, we can obtain our dynamic accumulation equation of the endogenous depreciation rate,

$$\dot{\Delta}_t = -a\left[\Delta_t - \delta\left(s_t\right)\right].$$

The hamiltonian of the representative agent is written as

$$H = u\left(c\right) + \lambda\left[f\left(k\right) - c - s - \Delta k\right] - \mu a\left[\Delta - \delta\left(s\right)\right],$$

where λ is the positive co-state variable with respect to the physical capital k, and μ is the negative co-state variable with respect to the depreciation rate Δ. Given the initial capital stock $k\left(0\right) = k_0$ and the initial depreciation rate $\Delta_0 = 0$, we can easily obtain the first-order conditions:

$$\lambda = u'\left(c\right), \tag{17}$$

$$\lambda = a\mu\delta'\left(s\right), \tag{18}$$

$$\dot{\lambda} = -\left[f'\left(k\right) - \Delta - \rho\right]\lambda, \tag{19}$$

$$\dot{u} = -\left[a\delta'\left(s\right)k + a + \rho\right]\mu, \tag{20}$$

$$\dot{k} = f\left(k\right) - c - s - \Delta k, \tag{21}$$

$$\dot{\Delta} = -a\left[\Delta - \delta\left(s\right)\right], \tag{22}$$

and the TVC: $\lim_{t\to\infty} e^{-\rho t}\lambda k = \lim_{t\to\infty} e^{-\rho t}\mu\Delta = 0$.

Equation (17) and (18) are the familiar intratemporal optimality conditions which mean that the marginal utility of consumption is equal to the marginal value of physical capital and the marginal value of expenditure on the decrease of the depreciation rate. Equation (19) and (20) are the Euler equations depicting the inter-temporal optimum. Equation (21) and (22) are the dynamic accumulation functions of the economy.

3.2 The Dynamic System

From (17) and (19), we obtain:

$$\dot{c} = -\frac{u'\left(c\right)}{u''\left(c\right)}\left[f'\left(k\right) - \Delta - \rho\right]. \tag{23}$$

Taking derivative on equation (13b) totally, we have?:

$$\frac{ds}{dc} = \frac{u''(c)}{a\delta''(s)\mu} > 0,$$
$$\frac{ds}{d\mu} = -\frac{\delta'(s)}{\delta''(s)\mu} < 0.$$

Hence, we can represent s as a function of c, μ, i.e., $s = s(c, \mu)$. Substituting it into (20), (21) and (22), we obtain the whole dynamics of the economic system:

$$\dot{c} = -\frac{u'(c)}{u''(c)}\left[f'(k) - \Delta - \rho\right], \tag{24a}$$

$$\dot{\mu} = -\mu\left[a\delta'(s(c,\mu))k + a + \rho\right], \tag{24b}$$

$$\dot{k} = f(k) - c - s(c,\mu) - \Delta k, \tag{24c}$$

$$\dot{\Delta} = -a\left[\Delta - \delta(s(c,\mu))\right]. \tag{24d}$$

3.3 The Steady State

Imposing the stability condition $\dot{c} = \dot{\mu} = \dot{k} = \dot{\Delta} = 0$, we can obtain the steady state $(c^*, \mu^*, k^*, \Delta^*)$ of the economy described by the following equations:

$$f'(k^*) = \Delta^* + \rho, \tag{25}$$

$$ak^*\delta'(s(c^*,\mu^*)) + a + \rho = 0, \tag{26}$$

$$f(k^*) = c^* + s(c^*,\mu^*) + \Delta^* k^*, \tag{27}$$

$$\Delta^* = \delta(s(c^*,\mu^*)). \tag{28}$$

Substituting (28) into (25) and (27), we get a group of equations:

$$f'(k^*) = \delta(s(c^*,\mu^*)) + \rho, \tag{29}$$

$$ak^*\delta'(s(c^*,\mu^*)) + a + \rho = 0, \tag{30}$$

$$f(k^*) = c^* + s(c^*,\mu^*) + \delta(s(c^*,\mu^*))k^*. \tag{31}$$

Totally differentiating (30), we get:

$$\frac{d\mu^*}{dk^*} = \frac{-\delta'(s(c^*,\mu^*))}{k^*\delta''(s(c^*,\mu^*))s_\mu(c^*,\mu^*)} < 0,$$
$$\frac{d\mu^*}{dc^*} = -\frac{s_c(c^*,\mu^*)}{s_\mu(c^*,\mu^*)} > 0.$$

Hence, we describe μ^* as a function of c^* and k^*, i.e., $\mu^* = \mu(c^*, k^*)$. Moreover, substituting it into (29) and (31) leads to:

$$f'(k^*) = \delta(s(c^*, \mu(c^*, k^*))) + \rho, \tag{32}$$

$$f(k^*) = c^* + s(c^*, \mu(c^*, k^*)) + \delta(s(c^*, \mu(c^*, k^*)))\, k^*, \tag{33}$$

which are the same as (12) essentially.

Next, we consider the existence of the equilibria. Totally differentiating (32) and (33), we obtain:

$$\frac{dc^*}{dk^*} = -\frac{f''(k^*) - \delta'(s(c^*, \mu(c^*, k^*)))\, s_\mu(c^*, \mu(c^*, k^*))\, \mu_k(c^*, k^*)}{\delta'(s(c^*, \mu(c^*, k^*)))\,[s_c(c^*, \mu(c^*, k^*)) + s_\mu(c^*, \mu(c^*, k^*))\, \mu_c(c^*, k^*)]},$$

$$\frac{dc^*}{dk^*} = \frac{-f'(k^*) + \delta(s(c^*, \mu(c^*, k^*))) + s_c(c^*, \mu(c^*, k^*))\,[\mu_k(c^*, k^*) + k^* s_\mu(c^*, \mu(c^*, k^*))\, \delta'(s(c^*, \mu(c^*, k^*)))]}{1 + s_c(c^*, \mu(c^*, k^*)) + s_\mu(c^*, \mu(c^*, k^*))\, \mu_c(c^*, k^*) + \delta'(s(c^*, \mu(c^*, k^*)))\, k^*\,[1 + s_\mu(c^*, \mu(c^*, k^*))\, \mu_c(c^*, k^*)]},$$

whose signs cannot be determined based on the present assumptions. So we can say nothing about the existence of the equilibria with respect to these general functional forms. In section 3.5, we can get some interesting numerical solutions with special specifications of the utility function, production function and depreciation function.

3.4 The Stability of the Extended Model

Similar to the simple model, we assume the equilibrium exists at first. Linearizing system (24) around the steady state leads to:

$$\begin{pmatrix} \dot{c} \\ \dot{\mu} \\ \dot{k} \\ \dot{\Delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{u'(c^*)}{u''(c^*)} f''(k^*) & \frac{u'(c^*)}{u''(c^*)} \\ a\mu^* k^* \delta''(s^*)\, s_c^* & a\mu^* k^* \delta''(s^*)\, s_\mu^* & a\mu^* \delta'(s^*) & 0 \\ -1 - s_c^* & -s_\mu^* & \rho & -k^* \\ a\delta'(s^*)\, s_c^* & a\delta'(s^*)\, s_\mu^* & 0 & -a \end{pmatrix} \begin{pmatrix} c - c^* \\ \mu - \mu^* \\ k - k^* \\ \Delta - \Delta^* \end{pmatrix}.$$

We know that there are two state variables in this system. Now, applying the theorems formulated in Dockner (1985, 1991), we can obtain the stability proposition of this four dimensional dynamic system through calculations of the Jacobian matrix of the correponding linearized system. However, we will leave out this calculation because it is the same as the simple model, and because the numerical results of the following subsection are more convicing.

3.5 Numerical Solutions

3.5.1 Example 1: An Explicit Solution with a Unique Steady state

We assume that $f(k) = Ak^{\alpha}$, $\delta(s) = e^{-s}$, and $u(c) = \log c$. With the same calculations as the aforementioned general case, we can obtain the unique steady state of the economy:

$$k^{*} = \left(\frac{a+\rho}{A\alpha a}\right)^{1/\alpha},$$
$$c^{*} = Ak^{*\alpha} - \log\left(\frac{ak^{*}}{a+\rho}\right) - \frac{a+\rho}{a},$$

with the related equilibrium maintenance expenditure: $s^{*} = \log(ak^{*}/a+\rho)$ and the equilibrium depreciation rate: $\Delta^{*} = a+\rho/ak^{*}$. However, the local stability of this unique equilibrium cannot be determined and it depends on specific parameter values.

Let $A = 2$, $\rho = 0.02$, $\alpha = 0.4$, and $a = 10$, the steady state of the endogenous variables are $c^{*} = 0.9421$, $k^{*} = 1.7557$, $\Delta^{*} = 0.5707$, and $s^{*} = 0.5609$. The eigenvalues of the Jacobian matrix are 14.6445, $-7.3082 \pm 6.9747i$, and 0.0118. Thus, the steady state is saddle-point stable.

If we change the value of ρ from 0.02 into 0 without any change of the other variables, we can obtain the steady state of the endogenous variables: $c^{*} = 0.9421$, $k^{*} = 1.7469$, $\Delta^{*} = 0.5724$, and $s^{*} = 0.5579$. The eigenvalues of the Jacob matrix are 14.5886, $-7.2901 \pm 6.9086i$, and -0.0084. Hence, the equilibrium is locally indeterminate.

3.5.2 Example 2: A Case with Multiple Equilibria

Let $f(k) = Ak^{\alpha}$, $u(c) = \log c$, and $\delta(s) = \frac{1}{1+s}$. If we take $A = 2.5$, $\alpha = 0.5$, $\rho = 0.04$, and $a = 10$, there exits a unique steady state: $k^{*} = 38.4406$, $c^{*} = 9.9911$, $\Delta^{*} = 0.1615$, and $s^{*} = 5.1939$. And the eigenvalues of the Jacobian matrix are 10.1217, -10.0817, 0.0946, and -0.0546. Hence the unique steady state is saddle-point stable.

If we take $A = 10$, $\alpha = 0.3$, $\rho = 0.04$, and $a = 10$, we obtain two steady states. One is

$$(c_1^{*}, k_1^{*}, \Delta_1^{*}, s_1^{*}) = (61.5861, 35.4540, 0.1276, 6.8398),$$

and the corresponding eigenvalues of the Jacobian matrix are 10.1057, -0.1558, 0.1958, and -10.0637. Thus, this steady state is locally saddle-point stable.

The other steady state is

$$(c_2^{*}, k_2^{*}, \Delta_2^{*}, s_2^{*}) = (1754.4188, 95.1915, 0.0239, 40.8440),$$

and the corresponding eigenvalues of the Jacobian matrix are 10.0520, -10.0120, 0.0010, and 0.0390. Therefore, this steady state is unstable.

Letting $A = 3$, $\alpha = 0.35$, $\rho = 0.02$, and $a = 10$, we obtain four steady states,

$$(c_1^*, k_1^*, \Delta_1^*, s_1^*) = (4.5652, 1.6341, 0.7831, 0.2770),$$
$$(c_2^*, k_2^*, \Delta_2^*, s_2^*) = (56.9091, 4231.8940, 0.7831, 0.2770),$$
$$(c_3^*, k_3^*, \Delta_3^*, s_3^*) = (4.1915, 1.1911, 0.9172, 0.0903),$$
$$(c_4^*, k_4^*, \Delta_4^*, s_4^*) = (34.5771, 984.6398, 0.0319, 30.3954).$$

and their corresponding eigenvalues of the Jacobian matrix are:

$$10.4667, 0.5272, -0.5072, -10.4467,$$
$$11.0789, 1.7348, -1.7148, -11.0589,$$
$$10.5607, 0.7007, -0.6807, -10.5407,$$
$$10.0359, -10.0159, 0.0100 \pm 0.0137i,$$

which imply that the first three steady states are saddle-point stable and the last one is unstable.

Now supposing $A = 2$, $\rho = 0.04$, $\alpha = 0.35$, and $a = 10$, we obtain four unstable steady states:

$$(c_1^*, k_1^*, \Delta_1^*, s_1^*) = (15.9923, 311.6698, 0.0567, 16.6365),$$
$$(c_2^*, k_2^*, \Delta_2^*, s_2^*) = (10.1675, 15.875, 0.2512, 2.9804),$$
$$(c_3^*, k_3^*, \Delta_3^*, s_3^*) = (4.4004, 4.5234, 0.4707, 1.1247),$$
$$(c_4^*, k_4^*, \Delta_4^*, s_4^*) = (23.3945, 978.4100, 0.0320, 0.0320),$$

with the corresponding eigenvalues of the Jacobian matrix as follows:

$$10.0683, -10.0283, 0.0200 \pm 0.0221i,$$
$$10.1675, -10.1275, 0.0200 \pm 0.1374i,$$
$$10.2851, -10.2451, 0.0200 \pm 0.2433i,$$
$$10.0560, -10.0160, 0.0081, 0.0319.$$

If we take $A = 0.5$, $\alpha = 0.35$, $\rho = 0.02$, and $a = 10$, we cannot find real roots of equilibrium capital stock. That is to say, no steady state exists.

4 Concluding Remarks

This paper explores the implications of the hypothesis that the depreciation rate of the physical capital of the economy is determined by the efforts of maintenance and repairs of the agent. By introducing two different endogenous mechanisms, we find that the convergence property of the unique steady state in the standard neoclassical growth model cannot be guaranteed. In fact, complex dynamics emerge, with multiple equilibria and indeterminacy.

In existing studies, many papers have derived multiple equilibria and indeterminacy by making strong assumptions about the utility function, the market structure, and the production technology. As we have seen, some papers put the state variable into the utility function; or introduce market imperfections such as monopolistic competition and externalities; or make strong assumptions on the production technology such as increasing return to scale. Different from the literature, we attach importance to the depreciate rate which is regarded as a constant in the standard neoclassical model and endogenize it in the standard model. Surprisingly, once the depreciation rate is endogenized, the uniqueness and convergence of the equilibrium do not guarantee, and multiple equilibria and indeterminacy emerge. Furthermore, this endogeneity of depreciation rate is supported by many empirical studies and accounting studies. Therefore, we obtain a new channel of reaching multiple equilibria and indeterminacy.

As another study by us (Luo, Wang & Zou, 2010), we introduce the endogenous depreciation into the theory of firm. With endogenous depreciation rate in the q-theory of investment, we find out that the standard equilibrium result of the q-theory should be revised. In our future study, we will study the effects of macroeconomic policies or different developmental strategies within this new framework. And we think that the new research will help improve our knowledgement on the theory of growth, investment and development.

References

[1] Auernheimer, Leonardo (1986), "Variable Depreciation and Some of Its Implications,". *The Canadian Journal of Economics* 19(1), 99-113.

[2] Baumol, William J. (1971), "Optimal Depreciation Policy: Pricing the Products of Durable Assets," *The Bell Journal of Economics and Management Science* 2(2), 638-656.

[3] Benhabib, Jess, Qinglai Meng and Kazuo Nishimura (2000), "Indeterminacy under Constant Return to Scale in Multisector Economies," *Econometrica* 68, 1541-1548.

[4] Benhabib, Jess and Roger E.A. Farmer (1994), "Indeterminacy and Increasing Returns," *Journal of Economic Theory* 63, 19-14.

[5] Bitros, George C. (1976), "A Statistical Theory of Expenditures in Capital Maintenance and Repair," *The Journal of Political Economy* 84(5), 917-936.

[6] Boldrin, Michele and Luigi Montrucchio (1986), "On the Indeterminacy of Capital Accumulation Paths," *Journal of Economic Theory* 40, 26-39.

[7] Dangl, Thomas and Franz Wirl (2004), "Investment under Uncertaity: Calculating the Value Function when the Bellman Equation Cannot Be Solved Analytically," *Journal of Economic Dynamic & Control* 28, 1437-1460.

[8] Dockner, Engelbert (1985), "Local Stability Analysis in Optimal Control Problems with Two State Variables.".In G. Feichtinger, eds., *Optimal Control Theory and Economic Analysis 2*. Amsterdam, North-Holland.

[9] Dockner, Engelbert and Gustav Feichtinger. (1991), "On the Optimality of Limit Cycles in Dynamic Economic Systems," *Journal of Economics* 53(1), 31-50.

[10] Evans, George W., Seppo Honkapohja and Paul Domer (1998), "Growth Cycles," *The American Economic Review* 88(3), 495-55.

[11] Fujisaki, Seiya and Kazuo Mino (2009), "Long-run Impacts of Inflation Tax with Endogeous Capital Depreciation," *GCOE Discussion Paper Series* No.69.

[12] Gylfason, Thorvaldur and Gylfi Zoega (2007), "A Golden Rule of Depreciation," *Economic Letters* 96, 357-362.

[13] Howitt, Peter and P. R. McAfee (1992), "Animal Spirits," *American Economic Review* 82,493-507.

[14] Howitt, Peter and P. R. McAfee. (1988), "Stability of Equilibrium with Trade Externalities," *Quarterly Journal of Economics* 103, 261-277

[15] Kehoe, T.J., D. Levine, and Paul Romer (1992), "Characterizing Equilibria of Optimization Problems with Externalities and Taxes as Solutions to Optimization Problems," *Economic Theory* 103, 695-714.

[16] Kurz, Mordecai (1968), "Optimal Economic Growth and Wealth Effects," *International Economic Review* 9(3), 348-357.

[17] Luo, Wang and Zou (2010), "The q-theory of investment and endogenous depreciation", forthcoming

[18] Matsuyama, Kiminori (1991), "IncreasingReturns,Industrialization,and Indeterminacy of Equilibrium," *The Quarterly Journal of Economics* 106(2), 617-650.

[19] Mukoyama, Toshihiko (2008), "Endogenous Depreciation, Mismeasurement of Aggregate Capital, and the Productivity Slowdown," *Journal of Macroeconomics* 30, 513-522.

[20] Murphy, K., A. Schleifer, and R. Vishny (1989), "Building Blocks of Market Clearing Bussiness Cycle Models," *NBER Macroeconomics Annual 1989*.

[21] Rioja, Felix K. (2003), "Filling Potholes: Macroeconomic Effects of Maintenance versus New Investments in Public Infrostructure," *Journal of Public Economics* 87,2281-2304.

[22] Spear, S. (1991), "Growth, Externalities and Sunspots," *Journal of Economic Theory* 215-223.