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Calls and Couples: Communication, Connections, Joint — Consumption and Transfer Prices*

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The article explores joint consumption equilibrium environments. It illustrates network formation through one-to-one directional synapses. Family (couple) arrangements, spontaneously generated under a decentralized general equilibrium price system are suggested — involving link and direction-specific transfer prices along with standard resource one. The research also inspects preference characteristics able to generate monogamous choices and assortative matching and mating. Assortative mating (and income pooling) is clarified, related to exclusivity or taste-for unicity at the utility level with respect to shared good, with optimal assignment connected to equalization of the marginal benefit of the match — adequately defined — across individuals in the economy.

Contrast with a multiple external effect good — one-to-many communication; (or) shared by a fixed number of, more than two, individuals; common property — and with a pure public good is also provided. If paired consumption with end-point specificity generates (or may generate), under reasonable assumptions, a unique decentralized equilibrium solution, supporting an efficient allocation, multiple agent sharing among more than two individuals and individual types requires, along with excludability, perfect differentiation of a larger number of consumption — partnership — roles.

Key Words: Shared goods; Joint consumption; Cost-sharing; Communications; Call; Linkage; Network nodes; Synapses; Matching; (Assortative) Mating; Couple goods; Family formation; Dowry; Transfer prices; Theory of the firm.

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1. INTRODUCTION

Mutual agreement is required for a large number of everyday transactions. Some are over a pure private good or service, and standard marginal pricing insures efficient allocations. Others, generate partial externalities or are even totally public, requiring superseding judgement. A fringe (...) are social in nature, its consumption implying benefits for two — or a given number of — affected agents. They may or may not require indirect costs from those traders (e.g., time) — they may or may not involve an externality —, they are identifiable both by the initiating and ending side of the consumption proposition and require complete consensus regarding its consumption/expenditure level.

The requirement of mutual agreement — involving excludability — allows a decentralized price system to insure an efficient allocation, provided discrimination between the two consumption sides is perfect: then, effectively, it is as if the two roles would distinguish themselves as two (times the number of individual types in the economy) different goods but not sold separately. The argument resembles the one applied to club goods — yet, here, the externality status is minor to qualify equilibrium properties[1], confined to a given or fixed number of people[2], and stresses the requirement of equal consumption — or sharing — of a total common "property" or durable; optimal pricing is (can be) achieved through transfers — or implicit consumption price discrimination —, which are due even if agents are homogeneous as long as they value differently the two roles (making and attending calls) in the "call society".

Understandably, a similar modelling framework has been applied in the economics of family and family formation: early examples[3] are Manser and Brown (1980) and McElroy and Horney (1981), suggesting marriage for allowing joint consumption by two agents — that bargain with each other while possessing, maintaining well-defined, "selfish"[4], individual preferences and budget constraints[5] — of special — household — public goods. Even if similar, our formalization presents a crucial difference: excludability by either side, and "family role" definition for each potential match; then,

[1] Or we could say that we would fall under Coase's theorem ...

[2] Say, total congestion is achieved with a fixed or maximum number of partners.

[3] That also include, more recently, Lam (1988) and Lundberg and Pollak (1993) — see Bergstrom (1996), Weiss (1997) and Vermeulen (2004) for recent surveys.

[4] Even if we can argue that some degree of altruism — and partner-specific inclination — can always be reflected in preferences for goods that are or must be shared with other individuals.

[5] Most of these family models end up by assuming pooled income.

under the usual ideal assumptions[6], a decentralized general equilibrium can be expected to promote efficient mating.

In family economics, two agent bargaining — interaction - is generally assumed. One can propose functional forms that are able to generate monogamy as polygamy — the later reproducing multi-(even if one-to-one)-connections. Assortative matching and mating can be studied with reference to the properties of the uncompensated individual demands and indirect utility functions[7] — which now also depend on partner(s) income and preferences — generated under exclusivity conditions. Then, transferable utility, or income — this mimicking, or effectively originating, budget pooling by the couple —, leads to the emergence of dowry systems. We then have a two-part tariff example[8] in the pricing system.

The framework can also encompass more complex societies — allow common property to be shared by more than two agents. In principle, network formation could be simulated by assuming that each connection between any two nodes is unique, with a node — as a neuron — having a life of its own. In the limit, joint-consumption by more than two individuals leads to a similar environment as that in the presence of a public good. With excludability, the only difficulty for a decentralized equilibrium arises from lack of competition and the leading (as others) role definition.

Also, productive factors — as outputs — can be shared by different divisions or plants of a firm... The theory suggests the adequate properties of an internal pricing scheme able to generate an efficient decentralized system management.

Finally, access cost and thus access pricing[9] is, however, only barely touched. Those (additional) costs of establishing n nodes, if a function of number of served clients — say, platform establishment costs—, would just require a (additional) fixed fee — suggesting a two-part tariff price of calls, as real immobile systems apply. We assume they are zero in most of the research — which may be reasonable for family study arrangements...). If these costs are periodic, a static model would apply; if not, the study of the subject could recommend a dynamic framework. The platform resembles a club good, but accessed by all individuals in the economy.

The exposition proceeds as follows: notation and individuals' utility functions are defined in section 2. Section 3 states the properties of an efficient allocation, and section 4 those of a decentralized equilibrium. In section

[6]Which, of course, rule out imperfect information or foresight, ex-post contract default, etc... The absence of the ideal conditions is what makes bargaining models of the family so appealing.

[7]See Becker (1973), Lam (1988). The analysis here differs both because budget constraints are never pooled, nor objective functions altered by connection establishment.

[8]See Littlechild (1975) for an early discussion of the subject.

[9]See Dewenter and Haucap, eds. (2007) for a recent overview of the subject.

5, we proceed to the derivation of demands, indirect utilities and equilibrium configurations for specific functional forms and in section 6, assortative mating is qualified under different transferability environments. Contrast with multiple emission entities is dealt with in section 7. The exposition ends with a brief summary.

2. NOTATION: PREFERENCES AND SHARED GOODS

There are n consumers in the economy. Each consumer, i, enjoys utility from the consumption of a private good, the quantity of which is denoted by x_i, from the quantity of "calls" he makes to individual j, z_i^j — the consumption of z proposed by i and accepted by j — and from those he receives from that same individual, y_i^j — the consumption of z proposed by j and accepted by i:

$$U^i(x_i, z_i^1, z_i^2, \ldots, z_i^{i-1}, z_i^{i+1}, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^{i-1}, y_i^{i+1}, \ldots, y_i^n), \quad (1)$$
$$i = 1, 2, \ldots, n$$

For simplicity, we will denote it by $U^i(x_i, z_i^j, y_i^j)$. Also,

$$\frac{\partial U^i(x_1, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial x_i} = U_x^i,$$

$\frac{\partial U^i(x_i, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial z_i^j} = U_{z_j}^i$ and $\frac{\partial U^i(x_i, z_i^1, z_i^2, \ldots, z_i^n, y_i^1, y_i^2, \ldots, y_i^n)}{\partial y_i^j} = U_{y_j}^i$. $U^i(x_i, z_i^j, y_i^j)$ is assumed to exhibit the usual properties — continuity, twice-differentiability and quasi-concavity.

The consumption of z requires feedback: it implies that:

$$z_i^j = y_j^i, \qquad i \neq j, i, j = 1, 2, \ldots, n \quad (2)$$

The distinction between z_i^j and y_j^i has two purposes: on the one hand, it represents the fact that there is perfect discrimination of the two consumption roles, and that i (may) faces a different net price for z_i^j than that charged to j for y_j^i; (but...) as we assume that there is mutual excludability between the i and j in the consumption of (both) z_i^j and y_j^i (z_j^i and y_i^j), i has the ability to control both z_i^j and y_i^j. These two conditions will allow for an efficient price system to develop. It would appear to apply well to calls, and it suggests the natural arising of gender differentiation — further stressed in economic dwelling by the requirement of definition of "head of household" status, of individual responsible for the child education...

On the other, it allows us to explore and understand similarities and differences between a pure externality (i.e., z_i^j and y_j^i are completely non-rival) and mere joint-consumption at equal levels — suggesting generalizations reproducing economies of scale in joint-consumption.

If i gets the same satisfaction from calling as from getting a call from j, then the utility has the special form:

$$U^i(x_i, z_i^1+y_i^1, z_i^2+y_i^2, \ldots, z_i^n+y_i^n) = U^i(x_i, z_i^j+y_i^j), \qquad i = 1, 2, \ldots, n \quad (3)$$

Also, if calls to and from any individual type are valued similarly, even if receiving and answering calls differentiated:

$$U^i(x_i, z_i^1 + z_i^2 + \cdots + z_i^{i-1} + z_i^{i+1} + \cdots + z_i^n, y_i^1 + y_i^2 + \cdots + y_i^{i-1} + y_i^{i+1} + \cdots + y_i^n)$$
$$= U^i(x_i, z_i + y_i) \quad i = 1, 2, \ldots, n \quad (4)$$

Of course, such additivity may occur in sets, with individual types arising distinctively for each i at the utility level.

Each individual is endowed with amount W_x^i of good x and W_z^i of good z. We will consider two scenarios:

- one in which only z_i^j requires W_z — on a one-to-one basis —, with y_i^j being a (almost) complete externality
- another in which both z_i^j as y_i^j require the use of W_z.

Yet, (2) — i.e., agreement from interlocutor —, must always be ensured. And, of course, whether an externality or pure joint-consumption at the same level for both sides applies (or other — see below), it must recognized by every individual in the economy.

A link between i and j requires no "fixed" costs, i.e., independent from the amount of z_i^j (or y_i^j) traded[10]. Network access (or set-up) costs — pure access to the markets where z and y are trade — are also assumed negligible[11].

A complex decentralized price system is proposed: p_x is the unit price of good x. The price of a call from i to j is composed of three parts: a general "call tariff" p_z, an answering tariff p_y, and a specific unit transfer from i to consumer j for attending the call, t_i^j. I.e., the consumption of z_i^j by i requires an additional "service" from j, priced at t_i^j.

[10]These could justify the emergence of monogamous couples even with preferences exhibiting taste for variety... And of dowries and bequests in the market independent of household quantity.

[11]They would not affect the general conclusions in what concerns marginal properties of interior solutions, provided that they are independent of network quantities aggregation... They would then justify an access pricing fixed fee independent of the use intensity. We will discuss their role briefly at the next sections.

Then the (exhausted) budget constraint of individual i is:

$$p_x x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j) z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i) y_i^j = p_x W_x^i + p_z' W_z^i \tag{5}$$

or

$$p_x x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j) z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i) z_j^i = p_x W_x^i + p_z' W_z^i \tag{6}$$

Summing (5) over i, as $\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i$, we conclude that the general tariffs must add up to the operating cost of a call p_z', at which W_z is traded.

$$p_z' = p_z + p_y \tag{7}$$

Notice that once we allow for transfers, payment can be collected on one-side of the call — charging $(p_z + p_y)$ to z — only: in practice, the actual individual transfers would also include the recovery of p_y.

For example, for common calls, $p_z = p_z'$ and $p_y = 0$. Child allowance schemes — see Lundberg and Pollak (1993), p. 1001 —, or merely nature's assignment of child-bearing and rearing costs, illustrate other unbalanced arrangements.

If y is non-rival with respect to z, p_z' is split between both sides of the call according to (7). Off-springs would appear to work as such. But a diner in a restaurant by a couple would involve twice the resources a solitary diner would — and (but) just require the same level of expenditure by the two individuals, the leveling of the quantity purchased by each of the two partners. In this type of cases, because now

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j = 2 \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i,$$

(aggregating (5)) $p_z' = \frac{p_z + p_y}{2}$ would price W_z — the average price paid by both i and j[12] — or rather $p_z + p_y$ would price one double unit of z_i^j-cum-y_j^i — given that it involves consumption duplication, nobody would want to buy or sell one of the two sides of the match separately. With joint-consumption, there will be a sort of sale complementarity; p_z' will then be the average price of the unit of W_z^i, sold in pairs.

A straight-forward generalization would allow for an intermediate state where $(z_i^j + y_j^i)\frac{1+\delta}{2}$ of W_z, $0 \leq \delta \leq 1$, is required to produce the "consumable" pair z_i^j-cum-y_j^i — purchased by i, $y_j^i \frac{1+\delta}{2}$ by j — a value of δ

[12] Allowing the price to still differ in both ends...

smaller than 1 representing economies of scale in household consumption; then $\frac{p_z+p_y}{1+\delta}$ would price W_z[13]. Or — allowing z_i^j to stand for half the total joint purchase so that $p'_z = \frac{p_z+p_y}{2}$ — assume utility functions are of the form $U^i(x_i, \frac{2z_i^j}{1+\delta}, \frac{2y_i^j}{1+\delta})$, requiring $z_i^j = y_j^i$, allowing or not differentiated pricing of z_i^j and y_j^i — hypothetically, δ could be pair specific, δ_i^j; such formulation would certainly be useful in the study of labor supply — if x_i denotes leisure, priced at W_i, $I^i = V^i + W_iT^i$ — full-income — where V^i and T^i are exogenous non-labor earnings and time endowment of i respectively, and pure private goods using W_z, g_{ij}, $j \neq i$, are also allowed such that we can write anybody's utility function as $U^i(x_i, g_{ij} + \frac{2z_i^j}{1+\delta}, \frac{2y_i^j}{1+\delta})$ or $U^i(x_i, \frac{g_{ij}}{2} + \frac{2z_i^j}{1+\delta}, \frac{g_{ij}}{2} + \frac{2y_i^j}{1+\delta})$ (and corner solutions naturally arise).

3. EFFICIENT ALLOCATION

Admit an efficient allocation is sought[14]. Then, one wants to maximize an individual's, say i, utility, subject to the existing endowments and limiting utility levels of all other consumers. Assume first that the receiver actually gets an externality. Then:

$$\max_{x_i, z_i^j, y_i^j, x_j, z_j^l, y_j^l} U^i(x_i, z_i^j, y_i^j) \tag{8}$$

s.t.:

$$U^j(x_j, z_j^l, y_j^l) \geq \overline{U}^j, j \neq i, j = 1, 2, \ldots, n \tag{8a}$$

$$z_i^j = y_j^i, i \neq j, i, j = 1, 2, \ldots, n \tag{8b}$$

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} W_x^i \tag{8c}$$

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j \leq \sum_{i=1}^{n} W_z^i \tag{8d}$$

[13]Of course, δ is assumed to be known by all market characters.

[14]The efficiency classification goes behond that of collective choice models — see Vermeulen (2004), for example —, once it is stated regardless of any pricing system... We characterize overall economic efficiency, not of intrahousehold allocation only.

In lagrangean form and replacing (8b):

$$\max_{x_i, z_i^j, x_j, z_j^l, \lambda_j, \mu_x, \mu_z} \quad U^i(x_i, z_i^j, z_j^i) + \sum_{\substack{j \neq i \\ j=1}}^{n} \lambda_j [\overline{U}^j - U^j(x_j, z_j^l, z_l^j)] \tag{9}$$

$$+ \quad \mu_x \left(\sum_{i=1}^{n} W_x^i - \sum_{i=1}^{n} x_i \right) + \mu_z \left(\sum_{i=1}^{n} W_z^i - \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j \right)$$

Interior FOC require:

$$\begin{aligned}
U_x^i - \mu_x &= 0 (1 \text{ equation}) && (10)\\
-\lambda_j U_x^j - \mu_x &= 0, j \neq i, j = 1, 2, \ldots, n (n-1 \text{ eqs.}) && (11)\\
U_{z_j}^i - \lambda_j U_{y_i}^j - \mu_z &= 0, j \neq i, j = 1, 2, \ldots, n (n-1 \text{ eqs.}) && (12)\\
U_{y_j}^i - \lambda_j U_{z_i}^j - \mu_z &= 0, j \neq i, j = 1, 2, \ldots, n (n-1 \text{ eqs.}) && (13)\\
-\lambda_j U_{z_l}^j - \lambda_l U_{y_j}^l - \mu_z &= 0, j \neq i, l \neq j, l = 1, 2, \ldots, n && (14)\\
-\lambda_j U_{y_l}^j - \lambda_l U_{z_j}^l - \mu_z &= 0, j \neq i, l \neq j, l = 1, 2, \ldots, n && (15)
\end{aligned}$$

along with (8a) (8c) and (8d) in equality. (12) to (15) include $n \times (n-1)$ different equations — the number of existing z_i^j's.

(10) and (11) imply the usual

$$\lambda_j = -\frac{U_x^i}{U_x^j}, j \neq i, j = 1, 2, \ldots, n \tag{16}$$

Replacing in (12) and (13) and equating the two (and (10)):

$$\frac{U_{z_j}^i}{U_x^i} + \frac{U_{y_i}^j}{U_x^j} = \frac{U_{y_j}^i}{U_x^i} + \frac{U_{z_i}^j}{U_x^j} (= \frac{\mu_z}{U_x^i}) = \frac{\mu_z}{\mu_x}, j \neq i, j = 1, 2, \ldots, n \tag{17}$$

Finally, from (14) and (15):

$$\frac{U_{z_l}^j}{U_x^j} + \frac{U_{y_j}^l}{U_x^l} = \frac{U_{y_l}^j}{U_x^j} + \frac{U_{z_j}^l}{U_x^l} (= \frac{\mu_z}{U_x^i}) = \frac{\mu_z}{\mu_x}, j \neq i, l \neq j, l = 1, 2, \ldots, n \tag{18}$$

If the second consumer does not obtain an externality, then (8d) is replaced by

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j \geq \sum_{i=1}^{n} W_z^i \tag{19}$$

The last term of the lagrangean (9) becomes

$$\mu_z \left(\sum_{i=1}^{n} W_z^i - 2 \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j \right).$$

Then (17) and (18) are replaced respectively by:

$$\frac{U_{z_j}^i}{U_x^i} + \frac{U_{y_i}^j}{U_x^j} = \frac{U_{y_j}^i}{U_x^i} + \frac{U_{z_i}^j}{U_x^j} (= 2\frac{\mu_z}{U_x^i}) = 2\frac{\mu_z}{\mu_x}, j \neq i, j = 1, 2, \ldots, n \qquad (20)$$

and

$$\frac{U_{z_l}^j}{U_x^j} + \frac{U_{y_j}^l}{U_x^l} = \frac{U_{y_l}^j}{U_x^j} + \frac{U_{z_j}^l}{U_x^l} (= 2\frac{\mu_z}{U_x^i}) = 2\frac{\mu_z}{\mu_x}, j \neq i, l \neq j, l = 1, 2, \ldots, n \quad (21)$$

(17) and (18) reproduce the well-known condition that the sums of the marginal rates of substitution of consumption partners must equate the marginal rate of transformation in the economy. (20) and (21) — in absence of externality —require that the average of those marginal rates of substitution equals the marginal rate of transformation.

Notice that the efficiency (Samuelson-type) condition, implying equalization of the sum (or averages if just joint-consumption) of the marginal rates of substitution between the shared and private good at the two consumption ends across the economy, is immune to mating or transferability considerations: it applies to any given welfare — ex-ante or ex-post transfers, as appropriate — utility levels of other individuals, $j \neq i$, we supply to the generic problem.

Suppose that in addition to the operating costs there is also a platform of size n, which costs in terms of $zC(n)$. If n is exogenously fixed, it would be clear that (19) would be replaced by

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y_i^j + C(n) \geq \sum_{i=1}^{n} W_z^i \qquad (22)$$

Samuelson condition would still be valid.

4. SUPPORTING GENERAL EQUILIBRIUM

Let each individual be subject to the general linear price conditions stated in section 2: in the economy, one unit of x costs p_x; one unit of z costs p'_z being jointly purchased and split between a caller and a receiver, accompanied by a consumer set/couple-specific unit transfer t_i^j. Any individual, i, solves:

$$\max_{x_i, z_i^j, y_i^j} U^i(x_i, z_i^j, y_i^j) \tag{23}$$

s.t.:

$$p_x x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j) + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i) y_i^j = p_x W_x^i + p_z' W_z^i = I^i \tag{24}$$

The lagrangean will take the form:

$$\max_{x_i, z_i^j, y_i^j, \mu} U^i(x_i, z_i^j, y_i^j) \tag{25}$$
$$+ \ \mu \left[p_x W_x^i + p_z' W_z^i - p_x x_i - \sum_{\substack{j \neq i \\ j=1}}^{n} (p_z + t_i^j) z_i^j - \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i) y_i^j \right]$$

and FOC for $i = 1, 2, \ldots, n$:

$$U_x^i - \mu p_x = 0 \tag{26}$$
$$U_{z_j}^i - \mu(p_z + t_i^j) = 0, j \neq i, j = 1, 2, \ldots, n \tag{27}$$
$$U_{y_j}^i - \mu(p_y - t_j^i) = 0, j \neq i, j = 1, 2, \ldots, n \tag{28}$$

with the budget constraint. Notice that as i can veto and ends up paying for y_j^i, optimization in it is due — and (28) arises — whether its consumption by i and j is completely non-rival (i.e., works as a complete "externality") or not: there is mutual excludability between the i and j in the consumption of (both) z_i^j and y_i^j. For a perfect externality, (28) would not take place — case that will be contrasted with the current one in section 7…

Then:

$$\frac{U_{z_j}^i}{U_x^i} = \frac{p_z + t_i^j}{p_x}, \quad j \neq i, j = 1, 2, \ldots, n (n - 1 \text{eqs. for each } i) \tag{29}$$

and

$$\frac{U_{y_j}^i}{U_x^i} = \frac{p_y - t_j^i}{p_x}, \quad j \neq i, j = 1, 2, \ldots, n (n - 1 \text{eqs. for each } i) \tag{30}$$

The conditions are valid for any consumer. Equilibrium requires additionally mutual consent on the call, (8b), with the price share, (7), that supplies and demands equate, i.e., (8c) and (8d) in equality.

$$z_i^j = y_j^i, \quad i \neq j, i, j = 1, 2, \ldots, n (n \times (n-1) \text{ eqs.}) \tag{31}$$

$$p'_z = p_z + p_y \tag{32}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} W_x^i \tag{33}$$

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i \tag{34}$$

It is straightforward to conclude that under common assumptions, provided we fix either $\frac{p_z}{p_x}$ or $\frac{p_y}{p_x}$, there will be an and a unique equilibrium relative price vector, $(\frac{p_z}{p_x}, \frac{p_y}{p_x}, \frac{p'_z}{p_x}, \frac{t_1^2}{p_x}, \ldots, \frac{t_1^n}{p_x}, \ldots, \frac{t_n^2}{p_x}, \ldots, \frac{t_n^{n-1}}{p_x})$ — with $n \times (n-1) + 3$ elements: we have $2(n-1)$ equations of form (29) and (30) and the budget constraint per consumer (generating the $n + 2n(n-1) = n(2n-1)$ individual demands), and the $n(n-1)+3$ composed of (31), (32) and aggregate market equilibrium ones — $n(3n-2)+3$ equations — yet, the sum of the budget constraints together with (33) and (34) imply (32) and only $n(3n-2)+2$ would be independent; on the other hand, the relative prices and the allocations z_i^j and y_j^i together include the same number of unknowns: $n \times (n-1) + 3$ relative prices and $n(2n-1)$ quantities.

In other words, the price system has now two degrees of freedom: not only (and as usual) may p_x be supplied, or x fixed as numeraire, as an exogenous convention about the splitting of the full price p'_z between the two "end-sides" of the deal — proposing and accepting parties — must also be agreed upon and supplied by society — usually taking the form $p_y = 0 \ldots$

One can show that such system supports an efficient solution. Every consumer j will solve a similar problem and choose baskets such that

$$\frac{U_{z_l}^j}{U_x^j} = \frac{p_z + t_j^l}{p_x}, \quad l \neq j, l = 1, 2, \ldots, n \tag{35}$$

and

$$\frac{U_{y_l}^j}{U_x^j} = \frac{p_y - t_l^j}{p_x}, \quad l \neq j, l = 1, 2, \ldots, n \tag{36}$$

Considering the relations towards $l = i$: (29) plus (36), and (30) plus (35) generate:

$$\frac{U_{z_j}^i}{U_x^i} + \frac{U_{y_i}^j}{U_x^j} = \frac{U_{y_j}^i}{U_x^i} + \frac{U_{z_i}^j}{U_x^j} = \frac{p_z + p_y}{p_x}, \quad j \neq i, j = 1, 2, \ldots, n \tag{37}$$

which reproduces (17), with $\frac{p_z+P_y}{p_x}$ having correspondence with $\frac{\mu_z}{\mu_x}$. As it must be valid for any consumer pair, it encompasses (18).

Then, effectively, unit transfers are set such that:

$$\frac{t_i^j}{p_x} = \frac{U_{z_j}^l}{U_x^i} - \frac{p_z}{p_x} = \frac{p_y}{p_x} - \frac{U_{y_i}^j}{U_x^j} \tag{38}$$

Notice that $t_i^j > 0$ and a transfer is due from i to j for the former's call if i appreciates (relative to consuming x) making calls to j more than its direct payment (i.e.,$\frac{p_z}{p_x}$); and if j appreciates (relative to consuming x) receiving calls from i less than people have to pay to receive calls (i.e., $\frac{p_y}{p_x}$).

No "lump-sum" transfers from i to j, are required or fit to insure equilibrium — a "dowry" would be here proportional to the bridal value: each link is free and everybody expected to be linked with everybody... They would be if there were (physical, i.e., in terms of the available resources, W_x and W_z) "fixed costs" associated with the establishment of each particular link.

However, once linkages are person-specific, the described equilibrium may be difficult to emerge due to lack of competition in unit transfer price formation; then, the exogeneity and constancy of the net of transfers prices as faced by individuals — required for (28) and (29) to apply — becomes questionable. One can claim that links are interchangeable, and/or that other links provide interpersonal-link comparisons — nevertheless, the argument remains...

Let us explore a little more deeply the demand formation in the economy.

Problem (25) generates conventional individual demands $x_i(I^i, p_x, p_z + t_i^1, p_z + t_i^2, \ldots, p_z + t_i^n, p_y - t_1^i, p_y - t_2^i, \ldots, p_y - t_n^i)$ $= x_i(\frac{I^i}{p_x}, 1, \frac{p_z+t_i^1}{p_x}, \frac{p_z+t_i^2}{p_x}, \ldots, \frac{p_z+t_i^n}{p_x}, \frac{p_y-t_1^i}{p_x}, \frac{p_y-t_2^i}{p_x}, \ldots, \frac{p_y-t_n^i}{p_x})$ and $z_i^j(I^i, p_x, p_z+t_i^1, p_z + t_i^2, \ldots, p_z + t_i^n, p_y - t_1^i, p_y - t_2^i, \ldots, p_y - t_n^i)$ $= z_i^j(\frac{I^i}{p_x}, 1, \frac{p_z+t_i^1}{p_x}, \frac{p_z+t_i^2}{p_x}, \ldots, \frac{p_z+t_i^n}{p_x}, \frac{p_y-t_1^i}{p_x}, \frac{p_y-t_2^i}{p_x}, \ldots, \frac{p_y-t_n^i}{p_x})$ — where $I^i = p_x W_x^i + p_z' W_z^i$ — enjoy standard properties. And z_i^j must equal $y_j^i(I^j, p_x, p_z+t_j^1, p_z + t_j^2, \ldots, p_z + t_j^n, p_y - t_1^j, p_y - t_2^j, \ldots, p_y - t_n^j)$ $= y_j^i(\frac{I^j}{n}, 1, \frac{p_z+t_j^1}{p_x}, \frac{p_z+t_j^2}{p_x}, \ldots, \frac{p_z+t_j^n}{p_x}, \frac{p_y-t_1^j}{p_x}, \frac{p_y-t_2^j}{p_x}, \ldots, \frac{p_y-t_n^j}{p_x})$, which is also a consumer demand, but of another individual.

Systems of Marshallian or uncompensated demands $x_i(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z+p_y)$ and $z_i^j(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z+p_y)$ independent of transfer prices can be derived from (37) and, replacing (35) and (36) in the budget constraint, from:

$$x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U^i_{z_j}}{U^i_x} z^j_i + \sum_{\substack{j \neq i \\ j=1}} \frac{U^i_{y_j}}{U^i_x} = \frac{I^i}{p_x} \tag{39}$$

$$= W^i_x + \frac{p_z + p_y}{p_x} W^i_z, i = 1, 2, \ldots, n$$

Those demand functions would be homogeneous of degree 0 in $I^1, I^2, \ldots, I^i, \ldots, I^n, p_x$ and $p_z + p_y$ but would not exhibit all of the other usual properties. They are independent of transfer prices because they already internalized its formation (rule). Moreover, each individual's demand — including that of the purely private good — is expected to be a function of everybody else's income, and not independent of its particular distribution, the same being true for indirect utility functions.

Compensated effects of an individual i's demand can be derived at fixed utility of all individuals, $x_i(U^1, U^2, \ldots, U^i, \ldots, U^n, p_x, p_z + p_y)$ — obeying (37) and $U^j(x_j, z^l_j, z^j_l) = U^j, j = 1, 2, \ldots, n$, and at fixed utility of i and fixed income of all others, $x_i(I^1, I^2, \ldots, U^i, \ldots, I^n, p_x, p_z + p_y)$.

Of equal relevance for private goods, demands conditional on the common purchases, $x_i(I^i, p_x, z^l_j, y^l_j) = x_i(I^i, p_x, z^l_j, z^j_l)$ would come from solving (39) with respect to x_i (with more private goods, it would also imbed equality of their common marginal rate of substitution to their relative prices) for individual i. For compensated demands, $x_i(U^i, p_x, z^l_j, y^l_j) = x_i(U^i, p_x, z^l_j, z^j_l)$ would arise then from the traditional conditions (here, just inverting the utility function; with more private goods, MRS between them should equal the corresponding price ratio), yet i's conditional expenditure function would be generated according to the left hand-side of (39).

Requiring the sum (over all i) of Marshallian demands $x_i(I^1, I^2, \ldots, I^i, \ldots, I^n, p_x, p_z + p_y) = x_i(\frac{I^1}{p_x}, \frac{I^2}{p_x}, \ldots, \frac{I^i}{p_x}, \ldots, \frac{I^n}{p_x}, 1, \frac{p_z+p_y}{p_x})$ to equalize available resource endowment (supply) in the economy — and replacing the I^i's by the corresponding definition — would allow us to infer the general equilibrium relative full price, $\frac{p_z+p_y}{p_x}$.

If the second consumer does not obtain an "externality", then (34) is replaced by

$$\sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} z^j_i + \sum_{i=1}^{n} \sum_{\substack{j \neq i \\ j=1}}^{n} y^j_i = \sum_{i=1}^{n} W^i_z \tag{40}$$

or, given (32):

$$2\sum_{i=1}^{n}\sum_{\substack{j\neq i\\ j=1}}^{n} z_i^j = \sum_{i=1}^{n} W_z^i$$

With the same preferences and endowments, the equilibrium allocation will differ from the one before, but share all other mathematical properties except for the optimal endowment price: now, $(p_z + p_y)$ is the price of a pair of units of W_z^i and (32) is (also) replaced by:

$$p_z' = \frac{1}{2}(p_z + p_y) \tag{41}$$

If consumers are homogeneous (have the same preferences and endowments) but receiving and making calls are valued differently so that the typical utility function is of type (4), there is only a need for two prices — potentially, p_z and p_y — to characterize equilibrium, yet z_i^j is sold to (in if there is no externality) pairs.

If form (3) is applicable — and there were indifference (perfect substitutability) between z_i^j and y_i^j at the utility level and at both consumption sides, as the marginal utility for i of consuming one extra unit of is equal to that of consuming y_i^j, the net price he will pay for either, say p_i^j, would equalize in an interior solution; then, simply adjusting z_i^j by not answering some, or prolonging a call by calling after a hang-up would insure an adequate distribution of expenses: choosing then z_i^j such that $p_i^j(z_i^j + y_i^j) = p_z' z_i^j$, would also insure that $p_z' z_j^i = p_j^i(z_j^i + y_j^i)$, both adding the full expenditure on the resource. Then, again, unit transfers are really redundant — the argument of potential lack of competition in unit transfer price formation removed — but, in general, not otherwise ...

With agent types multiplicity and some set additivity of form (4) at the utility level, the exogenous splitting rule of the total p_z' and perfect individual type identification — discrimination — and consumer replication, a uniquely decentralized equilibrium can arise, produce a unique equilibrium relative full price(s), a type-to-type specific transfer, and it is efficient. Then, it would be as if i buys z_i^j for $p_z + p_y$ and then j buys y_i^j from (individuals of type) i for $(p_y - t_j^i)$; replication — for competition — implies that some t_i^j's equalize.

Or, in a different light but representing the same structure, if we assume that n is a fixed number of possible connections, coinciding with the number of agent types in the economy, provided that calls with each type may accumulate — i.e., an individual of type i can receive calls from more than (as a fraction of those made by) one individual of type j —, the previous price system is sufficient. If they cannot, and only one individual of each type (that is, income and preferences, identifying i and j) can be connected to another to allow z_i^j, a lump-sum transfer system for each connection — with i receiving net $(K_i - K_j)$ from a connection with an individual of type $j, j \neq i, j = 1, 2, \ldots, n$ —, may emerge, leaving identical individuals indifferent in equilibrium.

Likewise, in family couples, (4) would hardly imply monogamy; if we allow for (3) and assume that there are fixed — n — individual types (characterized both by preferences and income level) in the economy and z_i^j represents a potential joint consumption of an individual of type i with another of type j, partner selection and stable family establishment could arise from extensive corner solutions, multiple marriages from less extensive ones. Gender (or "head of household" status) naturally distinguishes each side of the partnership and provides the required end-side discrimination — type identification should also be perfect —, and conditions for an efficient decentralized equilibrium are therefore staged.

A corner solution for $z_i^j = 0$ will require that also $y_i^j = 0$; it will occur iff, at the prevailing relative price level, $\frac{U^i_{z_j}}{U^i_x} + \frac{U^j_{y_i}}{U^j_x} < \frac{p_z + p_y}{p_x}$[15] at $z_i^j = y_j^i = 0$ at positive consumption of the other goods (and budget constraint multipliers in the appropriate lagrangean — according to Khun-Tucker conditions). If i and j are not connected, in the optimal solution, $z_i^j = y_j^i = 0$ and also $z_j^i = y_i^j = 0$. The equilibrium relative full price may be expected to go down while the inequality condition is not met as long as demand and supply allow, and exclusion — as in a purely private good does — would (could) occur spontaneously. For any interior solution, $U^i(x_i^*, z_i^{j^*}, y_i^{j^*}) > U^i(\frac{I^i}{p_x}, 0, 0)$; it must also supersede the utility that the individual can obtain paying in full any of the arguments other than x_i, say r, — consuming zero of the others — if shared consumption is allowed but not a psychological sine qua non. That is, for the solution for which (28), for $j = r$, is replaced by:

$$\frac{U^i_{z_r}}{U^i_x} = \frac{p_z + p_y}{p_x} \tag{42}$$

Or (30) by

$$\frac{U^i_{y_r}}{U^i_x} = \frac{p_z + p_y}{p_x} \tag{43}$$

(or both...) If marginal utilities are non-negative, these are the maximum individual net prices ever observed — a potential adoption by i of r's offspring.

[15] Corner solutions are commonly generated with linear functional forms — a special case of the CES.

If we impose exclusivity — or other exogenous discrete congestion threshold —, yet interchangeable connectivity (one can have but one mate, but any pair is possible ... Again, this may solve for the lack of competition in what transfer price formation is concerned ...), a more complex price exchange is required to insure equilibrium, now at the matching stage — which or may not feedback to the relative full price level of the shared resource in the economy. (Dowries are a type of transfer known in history, off-springs — involving expenditure — an obvious common good to parents.) Its study is deferred to section 6.

Finally, with platform establishment size costs $C(n)$ added to the problem, a fixed "lump-sum" fee, $C(n)/n$ would additionally be charged to each individual in the economy — expected if government provided or at least regulated to insure the provider with null (economic) profits. If there is no exclusivity, in theory, the platform is a public good. For example (39) would be replaced by:

$$x_i + \sum \begin{matrix} j \neq i \\ j = 1 \end{matrix}^{n} \frac{U^i_{z_j}}{U^i_x} z^j_i + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U^i_{y_j}}{U^i_x} + \frac{\frac{C(n)}{n}}{p_x} = \frac{I^i}{p_x} = W^i_x + \frac{p_z + p_y}{p_x} W^i_z,$$
$$i = 1, 2, \ldots, n \tag{44}$$

And even with excluclusivity of the couple assemblage type, these costs — to make known every partner possible — may occur ... Moreover, partial excludability does not alter that statement (we are not in the domain of a club good).

5. SPECIFIC FUNCTIONAL FORMS: MULTI-LEVEL CES UTILITY FUNCTIONS

In this section, we want to illustrate the impact of preferences on the network equilibrium formation. This is determined by utility function shapes and their, along with income, distribution; we therefore assume a general nested CES technology but allow individual specific characteristic coefficients.

We shall assume that individuals maximize utility subject to prices and an exogenous income $I^i = p_x W^i_x + p'_z W^i_z, I^i, p_x, p_z$, and p_y are externally fixed — replacing, for simplicity, the fixed individual endowments, p_x, and p_y (or p_z) of the previous section. An equilibrium will consist of individual allocations, a relative equilibrium full price, $\frac{p_z+p_y}{p_x}$, and net of unit (relative) transfer prices. For later convenience, we will present the marshallian demands and indirect utilities as a function of $I^i, i = 1, 2, \ldots, n, p_x$ and p'_z — say, applicable to a small economy that interconnects internally but takes international prices as given —, along with the autarky equilibrium price level — then replaced in demands and indirect utility.

Allocations can be determined from (37), $y_i^j = z_j^i$, and individual budget constraints (replaced by):

$$x_i + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U_{z_j}^i}{U_x^i} z_i^j + \sum_{\substack{j \neq i \\ j=1}}^{n} \frac{U_{y_j}^i}{U_x^i} = \frac{I^i}{p_x}, i = 1, 2, \ldots, n \tag{45}$$

Unit transfers can later be inferred from (38) — and net-of-transfers prices from (35) and (36) — but redundant to determine equilibrium.

For simplicity, let us consider an economy with a small number of consumers — let $n = 3$[16]. Utilities — that we assume separable in the set $[(x_i), (z_i^j, y_i^j), (z_i^{j'}, y_i^{j'})]$ — take the form:

$$\begin{aligned} & U^i(x_i, z_i^j, y_i^j, z_i^{j'}, y_i^{j'}) \\ = \; & A\left\{a_i x_i^{\rho_i} + a_{ij}[b_{ij} z_i^{j^{\lambda_{ij}}} + (1-b_{ij}) y_i^{j^{\lambda_{ij}}}]^{\frac{\rho_i}{\lambda_{ij}}} + a_{ij'}[b_{ij'} z_i^{j'^{\lambda_{ij'}}} + (1-b_{ij'}) y_i^{j'^{\lambda_{ij'}}}]^{\frac{\rho_i}{\lambda_{ij'}}}\right\}^{\frac{\mu_i}{\rho_i}} \\ & a_i + a_{ij} + a_{ij'} = 1, a_i, a_{ij}, a_{ij'} > 0, 0 < b_{ij}, b_{ij'} < 1, \rho_i, \lambda_{ij}, \lambda_{ij'} \leq 1 \end{aligned} \tag{46}$$

Then, $\sigma_i = \frac{1}{1-\rho_i}$ denotes the elasticity of substitution between (among ...) x_i and the two composites, $[b_{ik} z_i^{k^{\lambda_{ik}}}]^{\frac{1}{\lambda_{ik}}}$, in each of which $\sigma_{ik} = \frac{1}{1-\lambda_{ik}}$ is the elasticity of substitution between z_i^k and y_i^k within the composite $k = j, j'$.

(Even if we depart from this general functional form, we will only derive the full equilibrium for special cases. Features implied by some of the first-order optimization conditions are, nevertheless, inspected in general. . .)

The relevant ratios in the economy are then for $i = 1, 2, 3$ and $k = j, j'$:

$$\frac{U_{z_k}^i}{U_x^i} = \frac{a_{ik} b_{ik} [b_{ik} z_i^{k^{\lambda_{ik}}} + (1-b_{ik}) y_i^{k^{\lambda_{ik}}}]^{\frac{\rho_i}{\lambda_{ik}}-1} z_i^{k(\lambda_{ik}-1)}}{a_i x_i^{(\rho_i - 1)}} \tag{47}$$

and

$$\frac{U_{y_k}^i}{U_x^i} = \frac{a_{ik}(1-b_{ik})[b_{ik} z_i^{k^{\lambda_{ik}}} + (1-b_{ik}) y_i^{j^{\lambda_{ik}}}]^{\frac{\rho_i}{\lambda_{ik}}-1} y_i^{k(\lambda_{ik}-1)}}{a_i x_i^{(\rho_i - 1)}} \tag{48}$$

Given the strong separability, the ratios of marginal utilities of i with respect to j are independent of goods other than x_i and (z_i^j, y_i^j), i.e., of

[16]A competitive equilibrium would hardly be expected; but it allows us to derive explicit solutions highlighting the impact of preferences and income on the equilibrium.

$(z_i^{j'}, y_i^{j'})$. Yet, the general equilibrium system remains highly nonlinear; special cases for the link consumption sub-utility allow us to derive some conclusions:

i) $\lambda_{ik} = \rho_i, k = j, j'$: the sub-function embeds in the second-stage general CES formulation.

$$\frac{U_{z_k}^i}{U_x^i} = \frac{a_{ik} b_{ik} z_i^{k(\rho_i - 1)}}{a_i x_i^{(\rho_i - 1)}} \tag{49}$$

and

$$\frac{U_{y_k}^i}{U_x^i} = \frac{a_{ik}(1 - b_{ik}) y_i^{k(\rho_i - 1)}}{a_i x_i^{(\rho_i - 1)}} \tag{50}$$

Then:

$$\frac{a_{ik} b_{ik} z_i^{k(\rho_i - 1)}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ki}(1 - b_{ki}) z_i^{k(\rho_k - 1)}}{a_k x_k^{(\rho_k - 1)}} = \frac{p_z + p_y}{p_x}, i = 1, 2, 3; k = j, j' \tag{51}$$

A solution would be obtained combining the last expressions with the three budget constraints, leading to a nonlinear system:

$$p_x x_i + p_x \left[\frac{a_{ij} b_{ij} z_i^{j^{\rho_i}}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ij'} b_{ij'} z_i^{j'^{\rho}_i}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ij}(1 - b_{ij}) z_j^{i^{\rho_i}}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ij'}(1 - b_{ij'}) z_{j'}^{i^{\rho_i}}}{a_i x_i^{(\rho_i - 1)}} \right] = I^i \tag{52}$$

Reciprocity of some sort requires $a_{ik} b_{ik} = a_{ki}(1 - b_{ki})$. With reciprocity and constant ρ_i, (52) simplifies to:

$$p_x x_i + p_x \left\{ \left[1 + \left(\frac{a_{ji} b_{ji}}{a_{ij} b_{ij}} \right)^{\frac{1}{1-\rho}} \right] \frac{a_{ij} b_{ij} z_i^{j^{\rho}}}{a_i x_i^{(\rho - 1)}} + \left[1 + \left(\frac{a_{j'i} b_{j'i}}{a_{ij'} b_{ij'}} \right)^{\frac{1}{1-\rho}} \right] \frac{a_{ij'} b_{ij'} z_i^{j'^{\rho}}}{a_i x_i^{(\rho - 1)}} \right\} = I^i \tag{53}$$

Allow:

1) $\rho_3 = 1; \rho_1 = \rho_2 = \rho$ (but otherwise free parameters. Then:

$$\begin{aligned}
\frac{x_1}{z_3^1} &= \left[\left(\frac{p_z + p_y}{p_x} - \frac{a_{31} b_{31}}{a_3} \right) \frac{a_1}{a_{13}(1 - b_{13})} \right]^{\frac{1}{1-\rho}} \\
\frac{x_1}{z_1^3} &= \left[\left(\frac{p_z + p_y}{p_x} - \frac{a_{31}(1 - b_{31})}{a_3} \right) \frac{a_1}{a_{13} b_{13}} \right]^{\frac{1}{1-\rho}} \\
\frac{x_2}{z_3^2} &= \left[\left(\frac{p_z + p_y}{p_x} - \frac{a_{32} b_{32}}{a_3} \right) \frac{a_2}{a_{23}(1 - b_{23})} \right]^{\frac{1}{1-\rho}} \\
\frac{x_2}{z_2^3} &= \left[\left(\frac{p_z + p_y}{p_x} - \frac{a_{32}(1 - b_{32})}{a_3} \right) \frac{a_2}{a_{23} b_{23}} \right]^{\frac{1}{1-\rho}}
\end{aligned}$$

For $x_i > 0$, (for values of ρ such as 0) then $\frac{p_z+p_y}{p_x} > \frac{a_{3i}b_{3i}}{a_3}$ and $\frac{p_z+p_y}{p_x} > \frac{a_{3i}(1-b_{3i})}{a_3}, i = 1, 2$.

If $b_{3i} = b_{i3} = 0.5$, then $z_i^3 = z_3^i, i = 1, 2$.

The higher ρ (the higher the elasticity of substitution σ between the two composites for individuals 1 and 2), the lower the connections with 3 relative to the private good, i.e., the lower $\frac{z_3^i}{x_i}$ iff $\frac{p_z+p_y}{p_x} > \frac{a_{3i}b_{3i}}{a_3} + \frac{a_{i3}(1-b_{i3})}{a_i}$; and the lower $\frac{z_i^3}{x_i}$ iff $\frac{p_z+p_y}{p_x} > \frac{a_{i3}b_{i3}}{a_i} + \frac{a_{3i}(1-b_{3i})}{a_3}$.

2) $\rho_i = \rho_k = \rho$. (We have a regular CES). Reciprocity: $a_{ik}b_{ik} = a_{ki}(1 - b_{ki})$. Then:

Common elasticity of substitution requires:

$$z_i^k = \left\{ \frac{p_x}{p_z + p_y} \left[\frac{a_{ik}b_{ik}}{a_i} x_i^{(1-\rho)} + \frac{a_{ki}(1-b_{ki})}{a_k} x_k^{(1-\rho)} \right] \right\}^{\frac{1}{1-\rho}} \tag{54}$$

Reciprocity implies that, regardless of income:

$$z_i^k = \left\{ \frac{p_x}{p_z + p_y} a_{ik}b_{ik} \left[\frac{1}{a_i} x_i^{(1-\rho)} + \frac{1}{a_k} x_k^{(1-\rho)} \right] \right\}^{\frac{1}{1-\rho}} = z_k^i \left(\frac{a_{ik}b_{ik}}{a_{ki}b_{ki}} \right)^{\frac{1}{1-\rho}} \tag{55}$$

Assume further identical relative preferences for calls such that $\frac{a_{ik}b_{ik}}{a_i} = \frac{a_{ik}b_{ik}}{a_k} = \theta$, constant in the economy. Then:

$$\begin{aligned} z_i^k &= \left\{ \frac{p_x}{p_z + p_y} \theta [x_i^{(1-\rho)} + x_k^{(1-\rho)}] \right\}^{\frac{1}{1-\rho}} \\ &= z_k^i = x_i \left\{ \frac{p_x}{p_z + p_y} \theta \left[1 + \frac{x_k^{(1-\rho)}}{x_i^{(1-\rho)}} \right] \right\}^{\frac{1}{1-\rho}} \end{aligned} \tag{56}$$

For each consumer i — because $z_i^k = y_i^k - p_z + t_i^k = p_y - t_k^i = p_x \theta \frac{z_i^{k(\rho-1)}}{x_i^{(\rho-1)}} p_x x_i + (p_z + t_i^j) 2 z_i^j + (p_z + t_i^{j'}) 2 z_i^{j'} = I^i$. Then the three equations:

$$p_x x_i + 2 p_x \theta \left(\frac{p_x}{p_z + p_y} \theta \right)^{\frac{\rho}{1-\rho}} x_i^{(1-\rho)} \left\{ [x_i^{(1-\rho)} + x_j^{(1-\rho)}]^{\frac{\rho}{1-\rho}} + [x_i^{(1-\rho)} + x_{j'}^{(1-\rho)}]^{\frac{\rho}{1-\rho}} \right\} = I^i \tag{57}$$

or

$$p_x x_i \left[1 + 2\theta \left(\frac{p_x}{p_z + p_y} \theta \right)^{\frac{\rho}{1-\rho}} \left\{ \left[1 + \left(\frac{x_j}{x_i} \right)^{(1-\rho)} \right]^{\frac{\rho}{1-\rho}} + \left[1 + \left(\frac{x_{j'}}{x_i} \right)^{(1-\rho)} \right]^{\frac{\rho}{1-\rho}} \right\} \right] = I^i \tag{58}$$

allow us to retrieve the x_i's — the demands.

If income distribution is homogeneous, $x_i = x_k$ and

$$x_i = \frac{I_i}{p_x}\left[1+\theta^{\frac{1}{1-\rho}}\left(\frac{p_x}{p_z+p_y}\right)^{\frac{\rho}{1-\rho}} 2^{\frac{2-\rho}{1-\rho}}\right]^{-1} \tag{59}$$

and

$$z_i^k = \frac{I_i}{p_x}\left[\left(2\frac{p_x}{p_z+p_y}\theta\right)^{-\frac{1}{1-\rho}} + 2\left(\frac{p_x}{p_z+p_y}\right)^{-1}\right]^{-1} \tag{60}$$

$$\begin{aligned} v_i &= A[a_i x_i^\rho + (1-a_i) z_i^{k\rho}]^{\frac{\mu}{\rho}} &(61)\\ &= A\left(\frac{I_i}{p_x}\right)^\mu \left\{a_i\left[1+\theta^{\frac{1}{1-\rho}}\left(\frac{p_x}{p_z+p_y}\right)^{\frac{\rho}{1-\rho}} 2^{\frac{2-\rho}{1-\rho}}\right]^{-\rho}\right.\\ &+ \left.(1-a_i)\left[\left(2\frac{p_x}{p_z+p_y}\theta\right)^{-\frac{1}{1-\rho}} + 2\left(\frac{p_x}{p_z+p_y}\right)^{-1}\right]^{-\rho}\right\}^{\frac{\mu}{\rho}} \end{aligned}$$

Then, z_i^k — as $\frac{z_i^k}{x_i}$ — increases with ρ (and σ) iff $2\frac{p_x}{p_z+p_y}\theta > 1$ or $2\theta = 2\frac{a_{ik}b_{ik}}{a_i} > \frac{p_z+p_y}{p_x}$ — if the relative preference for the jointly consumed good is high. $\frac{\partial \nu_i}{\partial I^i} = \mu v_i \frac{1}{I^i} > 0$; as $\frac{\partial^2 \nu_i}{\partial I^{i2}} = (\mu-1)\mu v_i \frac{1}{I^{i2}}$, the whole economy "overly" rejoices — $\frac{\partial^2 \nu_i}{\partial I^{i2}} > 0$ — with an increase in everyone's endowment provided the utility function exhibits non-decreasing returns to scale. Also, the price of z is shared equally by any two partners:

$$p_z + t_i^j = p_y - t_j^i = \frac{p_z+p_y}{2} \tag{62}$$

Departing from (59) and summing both sides, multiplied by p, over the n individuals in the economy, equalizing to the total resource existence of x, we could solve for the general equilibrium relative full price level as:

$$\frac{p_z+p_y}{p_x} = 2^{(2-\rho)}\theta\left(\frac{\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l}\right)^{(1-\rho)} \tag{63}$$

(63) implies that the equilibrium relative price of z will decrease with the resource relative availability, $\frac{\sum_{l=1}^n w_z^l}{\sum_{l=1}^n w_x^l}$ ($n = 3$, the total number of individuals in the economy); and it will increase with the relative preference for the jointly consumed good, θ.

(63) could then be replaced in (59) to (62), using also the income definition, but there is not much insight to gain with that exercise.

Admit that income can differ across individuals but $\rho = 0$, i.e., of Cobb-Douglas format. Then, from (57), we conclude that individual demands are linear in income[17]:

$$p_x x_i + 4 p_x \theta x_i = I^i$$

This implies, on the one hand, the independence of the individual demand for the private good of income levels other than that of i itself; on the other — see (66) below —, and (also due to preference symmetry) the independence of the equilibrium relative full price of z of the income distribution in the economy.

$$x_i = \frac{I^i}{p_x}(1+4\theta)^{-1} \tag{64}$$

$$z_i^k = \frac{p_x}{p_z + p_y}\theta(x_i + x_k) = z_k^i = \frac{I^i + I^k}{p_z + p_y}\theta(1+4\theta)^{-1} \tag{65}$$

Replacing in the utility function, we obtain i's indirect utility function, v_i:

$$v_i = A(1+4\theta)^{-\mu_i}\left[\left(\frac{I^i}{p_x}\right)^{a_i}\left(\frac{I^i + I^j}{p_z + p_y}\theta\right)^{a_{ij}}\left(\frac{I^i + I^{j'}}{p_z + p_y}\theta\right)^{a_{ij'}}\right]^{\mu_i} \tag{66}$$

From (65), $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$ — there will be no assortative "matching" — nor positive, nor negative. $\frac{\partial \nu_i}{\partial I^j} = \mu_i a_{ij} v_i \frac{1}{(I^i + I^j)} > 0$; as $\frac{\partial^2 \nu_i}{\partial I^i \partial I^j} = \mu_i a_{ij} v_i \frac{\mu_i a_i (I^i+I^j)(I^i+I^{j'}) + \mu_i a_{ij'} I^i (I^i+I^j) - (1-\mu_i a_{ij}) I^i (I^i+I^{j'})}{(I^i+I^j)^2 (I^i+I^{j'}) I^i}$, (the equivalent to) positive assortative mating — subject explored in the next section — is expected — $\frac{\partial^2 \nu_i}{\partial I^i \partial I^j} > 0$ — with CRS or IRS ($\mu_i \geq 1$) at the utility level.

Also:

$$\frac{p_z + t_i^j}{p_x} = \frac{p_y - t_j^i}{p_x} = \theta \frac{x_i}{z_i^j} = \frac{I_i}{I_i + I_j}\frac{p_z + p_y}{p_x} \tag{67}$$

[17]Gorman polar forms — to which the Stone-Geary (and Cobb-Douglas), generating a linear expenditure system, subscribes — are known to generate public goods effects, or aggregate demands independent of individual income distributions (see Deaton and Muellbauer (1980), p. 144.) — because the form (quasi-homothetic utility function) implies linear individual Engel curves (exact aggregation also requires these to exhibit constant slopes across individuals — see Deaton and Muellbauer (1980), p. 150 —, satisfied then if individuals share common preferences). Quasi-linear functional forms — see Bergstrom and Cornes (1983), Lam (1988), Batina and Ihori (2005), p. 89 — are commonly used alternatives in public goods demand modelling for allowing (because the ratio of individual's marginal utilities of the public to the private good are linear and with constant slope across individuals in the latter) aggregation across individuals.

I pays a fraction of the price of the good(s) shared with j equal to the weight of his income relative to the pooled income of the two partners.

And given the Cobb-Douglas format of the utility, consuming something of all the goods is always worthwhile.

Internalizing equilibrium price formation in the Cobb-Douglas case:

$$\frac{p_z + p_y}{p_x} = 4\theta \frac{\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l} \tag{68}$$

The equilibrium relative price of z will decrease — here, being proportional to its inverse — with the resource relative availability, $\frac{\sum_{l=1}^n w_z^l}{\sum_{l=1}^n w_x^l}$; and it will increase with the relative preference for the jointly consumed good, θ. We can now replace them in the demands and indirect utility:

$$x_i = \frac{w_x^i \sum_{l=1}^n w_z^l + w_z^i 4\theta \sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l}(1+4\theta)^{-1} \tag{69}$$

$$z_i^k = z_k^i = \frac{(w_x^i + w_x^k)\sum_{l=1}^n w_z^l + (w_z^i + w_z^k)4\theta \sum_{l=1}^n w_x^l}{4\sum_{l=1}^n w_x^l}(1+4\theta)^{-1} \tag{70}$$

$$v_i^k = A(1+4\theta)^{-\mu_i} \tag{71}$$
$$\left\{ \left(\frac{w_x^i \sum_{l=1}^n w_z^l + w_z^i 4\theta \sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l} \right)^{a_i} \left[\frac{(w_x^i + w_x^k)\sum_{l=1}^n w_z^l + (w_z^i + w_z^k)4\theta \sum_{l=1}^n w_x^l}{4\sum_{l=1}^n w_x^l} \right]^{(1-a_i)} \right\}^{\mu_i}$$

$$\begin{aligned} \frac{p_z + t_i^j}{p_x} &= \frac{p_y - t_j^i}{p_x} \qquad (72) \\ &= \frac{w_x^i \sum_{l=1}^n w_z^l + w_z^i 4\theta \sum_{l=1}^n w_x^l}{(w_x^i + w_x^k)\sum_{l=1}^n w_z^l + (w_z^i + w_z^k)4\theta \sum_{l=1}^n w_x^l} \frac{p_z + p_y}{p_x} \\ &= \frac{w_x^i \sum_{l=1}^n w_z^l + w_z^i 4\theta \sum_{l=1}^n w_x^l}{(w_x^i + w_x^k)\sum_{l=1}^n w_z^l + (w_z^i + w_z^k)4\theta \sum_{l=1}^n w_x^l} 4\theta \frac{\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l} \end{aligned}$$

With fixed coefficient technologies — ρ tends to $-\infty$ —, $x_i = z_i^k = z_k^i = \frac{I^1+I^2+I^3}{3p_x+6(p_z+p_y)}$ and $v_i = A\left[\frac{I^1+I^2+I^3}{3p_x+6(p_z+p_y)}\right]^\mu$. With perfect substitutability — ρ tends to 1 —, consumption pairs could be expected.

ii) $\lambda_{ik} = 0, k = j, j'$: the sub-function is of the Cobb-Douglas type:

$$\begin{aligned} \frac{U_{z_k}^i}{U_x^i} &= \frac{a_{ik} b_{ik} [z_i^{k^{b_{ik}}} y_i^{k^{(1-b_{ik})}}]^{(\rho_i-1)} z_i^{k^{(b_{ik}-1)}}}{a_i x_i^{(\rho_i-1)}} \qquad (73) \\ &= \frac{a_{ik} b_{ik} y_i^{k^{[(1-b_{ik})(\rho_i-1)]}} z_i^{k^{(\rho_i b_{ik}-1)}}}{a_i x_i^{(\rho_i-1)}} \end{aligned}$$

and

$$\frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}(1-b_{ik})[z_i^{k^{b_{ik}}} y_i^{k^{(1-b_{ik})}}]^{(\rho_i-1)} y_i^{k^{-b_{ik}}}}{a_i x_i^{(\rho_i-1)}} \tag{74}$$
$$= \frac{a_{ik}(1-b_{ik}) z_i^{k^{[b_{ik}(\rho_i-1)]}} y_i^{k^{[\rho_i(1-b_{ik})-1]}}}{a_i x_i^{(\rho_i-1)}}$$

Then:

$$\frac{a_{ik} b_{ik} z_k^{i^{[(1-b_{ik})(\rho_i-1)]}} z_i^{k^{(\rho_i b_{ik}-1)}}}{a_i x_i^{(\rho_i-1)}} + \frac{a_{ki}(1-b_{ki}) z_k^{i^{[b_{ki}(\rho_k-1)]}} z_i^{k^{[\rho_k(1-b_{ki})-1]}}}{a_k x_k^{(\rho_k-1)}}$$
$$= \frac{p_z + p_y}{p_x}, i = 1, 2, 3; k = j, j'$$

iii) $\lambda_{ik} = 1, k = j, j'$: the sub-function is linear in the arguments.

$$\frac{U^i_{z_k}}{U^i_x} = \frac{a_{ik} b_{ik} [b_{ik} z_i^k + (1-b_{ik}) y_i^k]^{(\rho_i-1)}}{a_i x_i^{(\rho_i-1)}} = g_{ik} \tag{75}$$

and

$$\frac{U^i_{y_k}}{U^i_x} = \frac{a_{ik}(1-b_{ik})[b_{ik} z_i^k + (1-b_{ik}) y_i^k]^{(\rho_i-1)}}{a_i x_i^{(\rho_i-1)}} = \frac{1-b_{ik}}{b_{ik}} g_{ik} \tag{76}$$

For interior solutions to be possible:

$$g_{ik} + \frac{1-b_{ki}}{b_{ki}} g_{ki} = \frac{p_z + p_y}{p_x} \text{ and } g_{ki} + \frac{b_{ik}}{1-b_{ik}} g_{ik} = \frac{p_z + p_y}{p_x}$$

If $b_{ik} = 0.5, z_i^k > 0$ and $y_i^k = z_k^i = 0$ iff $b_{ki} < 0.5$; $z_i^k = 0$ and $y_i^k = z_k^i > 0$ iff $b_{ki} > 0.5$. If $z_i^k > 0$ and $y_i^k = z_k^i = 0$:

$$\frac{a_{ik} b_{ik}^{\rho_i} z_i^{k^{(\rho_i-1)}}}{a_i x_i^{(\rho_i-1)}} + \frac{a_{ki}(1-b_{ki})^{\rho_k} z_i^{k^{(\rho_k-1)}}}{a_k x_k^{(\rho_k-1)}} = \frac{p_z + p_y}{p_x} \tag{77}$$
$$\frac{a_{ki} b_{ki} (1-b_{ki})^{(\rho_k-1)} z_i^{k^{(\rho_k-1)}}}{a_k x_k^{(\rho_k-1)}} + \frac{a_{ik}(1-b_{ik}) b_{ik}^{(\rho_i-1)} z_i^{k^{(\rho_i-1)}}}{a_i x_i^{(\rho_i-1)}} < \frac{p_z + p_y}{p_x}$$

If $b_{ik} = 0.5$ for all i, k, we fall under (3) and there will be multiple values of z_i^k and y_i^k but a unique total $(z_i^k + y_i^k) = (z_k^i + y_k^i)$ satisfying equilibrium, including the corners represented by (77) in equality.

Admit a constant $\theta = \frac{a_{ik} b_{ik}^{\rho_i}}{a_i} = \frac{a_{ki}(1-b_{ki})^{\rho_k}}{a_k}$ for "active" links and $\rho_i = \rho$ for all i. Connections with all individuals require:

$$p_x x_i + p_x \theta \left(\frac{p_x}{p_z + p_y} \theta \right)^{\frac{\rho}{1-\rho}} x_i \left\{ \left[1 + \left(\frac{x_j}{x_i} \right)^{(1-\rho)} \right]^{\frac{\rho}{1-\rho}} + \left[1 + \left(\frac{x_{j'}}{x_i} \right)^{(1-\rho)} \right]^{\frac{\rho}{1-\rho}} \right\} = I^i \tag{78}$$

Then, we reached a similar expression to (57). Demands will be similar.

iv) $\lambda_{ik} = -\infty, k = j, j'$ and the sub-function is of the fixed coefficient, Leontief, type — $[b_{ik} z_i^{k^{\lambda_{ik}}} + (1 - b_{ik}) y_i^{j^{\lambda_{ik}}}]^{\frac{1}{\lambda_{ik}}}$ tends to $\min(z_i^k, y_i^k)$. Then, at efficient consumption levels, both items equalize and:

$$\frac{U_{z_k}^i}{U_x^i} = \frac{a_{ik} \min(z_i^k, y_i^k)^{(\rho_i - 1)}}{a_i x_i^{(\rho_i - 1)}} = \frac{a_{ik} z_i^{k^{(\rho_i - 1)}}}{a_i x_i^{(\rho_i - 1)}} \tag{79}$$

and

$$\frac{U_{y_k}^i}{U_x^i} = \frac{a_{ik} \min(z_i^k, y_i^k)^{(\rho_i - 1)}}{a_i x_i^{(\rho_i - 1)}} = \frac{a_{ik} y_i^{k^{(\rho_i - 1)}}}{a_i x_i^{(\rho_i - 1)}} \tag{80}$$

$z_i^k = y_i^k$: there is perfect complementarity between calls made or received by i from each k.

For interior solutions:

$$\frac{a_{ik} z_i^{k^{(\rho_i - 1)}}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ki} z_i^{k^{(\rho_k - 1)}}}{a_k x_k^{(\rho_k - 1)}} = \frac{p_z + p_y}{p_x}, i = 1, 2, 3; k = j, j' \tag{81}$$

with half of the conditions (compatible and) redundant, and

$$p_x x_i + 2 p_x \left[\frac{a_{ij} z_i^{j^{\rho_i}}}{a_i x_i^{(\rho_i - 1)}} + \frac{a_{ij'} z_i^{j'^{\rho_i}}}{a_i x_i^{(\rho_i - 1)}} \right] = I^i, i = 1, 2, 3 \tag{82}$$

For special cases, we arrive at solutions with similar properties as before.

Other interesting formulations would allow for a different degree of substitution between the two composites, say:

$$U^i(x_i, z_i^j, y_i^j, z_i^{j'}, y_i^{j'}) = A(a_i x_i^{\rho_i} + (1 - a_i) \tag{83}$$

$$\{a_{ij} [b_{ij} z_i^{j^{\lambda_{ij}}} + (1 - b_{ij}) y_i^{j^{\lambda_{ij}}}]^{\frac{\theta_i}{\lambda_{ij}}} + a_{ij'} [b_{ij'} z_i^{j'^{\lambda_{ij'}}} + (1 - b_{ij'}) y_i^{j'^{\lambda_{ij'}}}]^{\frac{\theta_i}{\lambda_{ij'}}} \}^{\frac{\rho_i}{\theta_i}})^{\frac{\mu_i}{\rho_i}}$$

$$0 < a_i, a_{ij}, a_{ij'}, b_{ij}, b_{ij'} < 1, a_{ij} + a_{ij'} = 1, \rho_i, \theta_i, \lambda_{ij}, \lambda_{ij'} \leq 1$$

FOC require:

$$\frac{U^i_{z_k}}{U^i_x} = (1-a_i)\frac{a_{ik}b_{ik}[b_{ik}z_i^{k^{\lambda_{ik}}} + (1-b_{ik})y_i^{k^{\lambda_{ik}}}]^{\frac{\theta_i}{\lambda_{ik}}-1} z_i^{k^{(\lambda_{ik}-1)}}}{a_i x_i^{(\rho_i-1)}} \tag{84}$$
$$\{a_{ij}[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})y_i^{j^{\lambda_{ij}}}]^{\frac{\theta_i}{\lambda_{ij}}} + a_{ij'}[b_{ij'}z_i^{j'^{\lambda_{ij'}}}]^{\frac{\theta_i}{\lambda_{ij'}}}\}^{\frac{\rho_i}{\theta_i}-1} = \frac{p_z + t_i^j}{p_x}$$

and

$$\frac{U^i_{y_k}}{U^i_x} = (1-a_i)\frac{a_{ik}(1-b_{ik})[b_{ik}z_i^{k^{\lambda_{ik}}} + (1-b_{ik})y_i^{k^{\lambda_{ik}}}]^{\frac{\theta_i}{\lambda_{ik}}-1} y_i^{k^{(\lambda_{ik}-1)}}}{a_i x_i^{(\rho_i-1)}} \tag{85}$$
$$\{a_{ij}[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})y_i^{j^{\lambda_{ij}}}]^{\frac{\theta_i}{\lambda_{ij}}} + a_{ij'}[b_{ij'}z_i^{j'^{\lambda_{ij'}}} + (1-b_{ij'})y_i^{j'^{\lambda_{ij'}}}]^{\frac{\theta_i}{\lambda_{ij'}}}\}^{\frac{\rho_i}{\theta_i}-1}$$
$$= \frac{p_y - t_j^i}{p_x}$$

Monogamous family formation can then be adequately modeled with reference to the threshold value of $\theta_i = 1$ or larger — representing taste for unicity...

In the limiting case where θ_i tends to $+\infty$, $\{a_{ij}[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})y_i^{j^{\lambda_{ij}}}]^{\frac{\theta_i}{\lambda_{ij}}} + (1-a_{ij})[b_{ij'}z_i^{j'^{\lambda_{ij'}}} + (1-b_{ij'})y_i^{j'^{\lambda_{ij'}}}]^{\frac{\theta_i}{\lambda_{ij'}}}\}^{\frac{1}{\theta_i}}$ tends to $\max\{[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})y_i^{j^{\lambda_{ij}}}]^{\frac{1}{\lambda_{ij}}}, [b_{ij'}z_i^{j'^{\lambda_{ij'}}} + (1-b_{ij'})y_i^{j'^{\lambda_{ij'}}}]^{\frac{1}{\lambda_{ij'}}}\}$ — note that $\min(x,y,z) = \max(x^{-1},y^{-1},z^{-1})^{-1}$ as well as $\min(x^{-1},y^{-1},z^{-1})^{-1} = \max(x,y,z)$ and use the fact that the CES tends to Leontief — and only pair-wise connections are formed. (Provided that SOC can still apply). Let us then consider such limiting case.

With three individual types, only 1 pair will be formed, let us say i and j. Then:

$$(1-a_i)\frac{b_{ij}[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})z_j^{i^{\lambda_{ij}}}]^{\frac{1-\lambda_{ij}}{\lambda_{ij}}} z_i^{j^{(\lambda_{ij}-1)}}}{a_i x_i^{(\rho_i-1)}} \tag{86}$$
$$\{[b_{ij}z_i^{j^{\lambda_{ij}}} + (1-b_{ij})z_j^{i^{\lambda_{ij}}}]^{\frac{1}{\lambda_{ij}}}\}^{(\rho_i-1)} +$$
$$(1-a_j)\frac{(1-b_{ji})[b_{ji}z_j^{i^{\lambda_{ji}}} + (1-b_{ji})z_i^{j^{\lambda_{ji}}}]^{\frac{1-\lambda_{ji}}{\lambda_{ji}}} z_i^{j^{(\lambda_{ji}-1)}}}{a_j x_j^{(\rho_j-1)}}$$
$$\{[b_{ji}z_j^{i^{\lambda_{ji}}} + (1-b_{ji})z_i^{j^{\lambda_{ji}}}]^{\frac{1}{\lambda_{ji}}}\}^{(\rho_j-1)} = \frac{p_z + p_y}{p_x}$$

or

$$(1-a_i)\frac{b_{ij}[b_{ij}z_i^{j^{\lambda_{ij}}}+(1-b_{ij})z_j^{i^{\lambda_{ij}}}]^{\frac{\rho_i-\lambda_{ij}}{\lambda_{ij}}}z_i^{j^{(\lambda_{ij}-1)}}}{a_i x_i^{(\rho_i-1)}} \tag{87}$$

$$+\ (1-a_j)\frac{(1-b_{ji})[b_{ji}z_j^{i^{\lambda_{ji}}}+(1-b_{ji})z_i^{j^{\lambda_{ji}}}]^{\frac{\rho_j-\lambda_{ji}}{\lambda_{ji}}}z_i^{j^{(\lambda_{ji}-1)}}}{a_j x_j^{(\rho_j-1)}}=\frac{p_z+p_y}{p_x}$$

j' either may consume only x, and $x_{j'}=\frac{I^{j'}}{p_x}$ — that occurring if $\rho_{j'}$ is large (certainly larger than 0). Or, he will pay his connections to only one of the other k's — either to i or to j, for whom the marginal utility of consumption of joint goods with j' is 0 — in full so that:

$$(1-a_{j'})\frac{b_{j'k}[b_{j'k}z_{j'}^{k^{\lambda_{j'k}}}+(1-b_{j'k})z_k^{j'^{\lambda_{j'k}}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}}}z_{j'}^{k^{(\lambda_{j'k}-1)}}}{a_{j'}x_{j'}^{(\rho_{j'}-1)}}=\frac{p_z+p_y}{p_x} \tag{88}$$

and

$$(1-a_{j'})\frac{(1-b_{j'k})[b_{j'k}z_{j'}^{k^{\lambda_{j'k}}}+(1-b_{j'k})z_k^{j'^{\lambda_{j'k}}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}}}z_k^{j'^{(\lambda_{j'k}-1)}}}{a_{j'}x_{j'}^{(\rho_{j'}-1)}}=\frac{p_z+p_y}{p_x} \tag{89}$$

Then $b_{j'k}z_{j'}^{k^{(\lambda_{j'k}-1)}}=(1-b_{j'k})z_k^{j'^{(\lambda_{j'k}-1)}}$ or $b_{j'k}^{\frac{\lambda_{j'k}}{\lambda_{j'k}-1}}=(1-b_{j'k})^{\frac{\lambda_{j'k}}{\lambda_{j'k}-1}}z_k^{j'^{\lambda_{j'k}}}$ and $[b_{j'k}z_{j'}^{k^{\lambda_{j'k}}}+(1-b_{j'k})z_k^{j'^{\lambda_{j'k}}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}}}$ $=[b_{j'k}+(1-b_{j'k})^{\frac{1}{1-\lambda_{j'k}}}b_{j'k}^{\frac{\lambda_{j'k}}{\lambda_{j'k}-1}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}}}z_{j'}^{k^{(\rho_{j'}-\lambda_{j'k})}}$. The expression becomes

$$(1-a_{j'})[b_{j'k}+(1-b_{j'k})^{\frac{1}{1-\lambda_{j'k}}}b_{j'k}^{\frac{\lambda_{j'k}}{\lambda_{j'k}-1}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}}}\frac{b_{j'k}z_{j'}^{k^{(\rho_{j'}-1)}}}{a_{j'}x_{j'}^{(\rho_{j'}-1)}}=\frac{p_z+p_y}{p_x} \tag{90}$$

His budget constraint becomes:

$$p_x x_{j'}+2(p_z+p_y)[1+\left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{\lambda_{j'k}-1}}]z_{j'}^{k}=I^{j'} \tag{91}$$

$$=\ p_x x_{j'}\{1+2[1+\left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{\lambda_{j'k}-1}}]\left(\frac{p_z+p_y}{p_x}\right)^{\frac{\rho_{j'}}{\rho_{j'}-1}}$$

$$\left[\frac{(1-a_{j'})b_{j'k}}{a_{j'}}\right]^{\frac{1}{1-\rho_{j'}}}[b_{j'k}+(1-b_{j'k})^{\frac{1}{1-\lambda_{j'k}}}b_{j'k}^{\frac{\lambda_{j'k}}{\lambda_{j'k}-1}}]^{\frac{\rho_{j'}-\lambda_{j'k}}{\lambda_{j'k}(1-\rho_{j'})}}\}$$

An equilibrium may then arise in which any of the three individuals pays its connections in full to one and only one individual, "free-riding" on the connections with other(s) — eventually, with an individual not paying.

In sum, with taste for unicity, a mating equilibrium mechanism must additionally arise ...

Consider $\lambda_{ik} = \rho_i$. Then, for the pair i, j, we fall back into

$$(1-a_i)\frac{b_{ij} z_i^{j(\rho_i-1)}}{a_i x_i^{(\rho_i-1)}} + (1-a_j)\frac{(1-b_{ji}) z_i^{j(\rho_j-1)}}{a_j x_j^{(\rho_j-1)}} = \frac{p_z+p_y}{p_x} \tag{92}$$

$$(1-a_j)\frac{b_{ji} z_j^{i(\rho_j-1)}}{a_j x_j^{(\rho_j-1)}} + (1-a_i)\frac{(1-b_{ij}) z_j^{i(\rho_i-1)}}{a_i x_i^{(\rho_i-1)}} = \frac{p_z+p_y}{p_x} \tag{93}$$

Budget constraints require for the pair i, j:

$$p_x x_i + p_x(1-a_i)[\frac{b_{ij} z_i^{j\rho_i}}{a_i x_i^{(\rho_i-1)}} + \frac{(1-b_{ij}) z_j^{i\rho_i}}{a_i x_i^{(\rho_i-1)}}] = I^i \tag{94}$$

Let reciprocity of some sort require $b_{ij} = (1-b_{ji})$. The traits of the general solution of (49) but now for two agents only are recovered.

For single payers:

$$I^{j'} = p_x x_{j'} \tag{95}$$
$$\left\{1+2\left[1+\left(\frac{b_{j'k}}{1-b_{j'k}}\right)^{\frac{1}{\rho_{j'}-1}}\right]\left(\frac{p_z+p_y}{p_x}\right)^{\frac{\rho_{j'}}{\rho_{j'}-1}}\left(\frac{(1-a_{j'})b_{j'k}}{a_{j'}}\right)^{\frac{1}{1-\rho_{j'}}}\right\}$$

If we allow for agent multiplicity, interior pairs can be formed only. Monogamy would be the rule against polygamy with perfect taste for unicity. Now, mating assorting can be studied not through interior consumption — z_i^j and y_i^j, more adequately qualifying "matching" —, but from corner solutions patterns — inspecting indirect utility functions properties.

In the symmetric preferences, Cobb-Douglas case ($\rho_i = 0$) for a (mated) individual i:

$$p_x x_i + 2p_x\theta x_i = I^i$$

where $\theta = \frac{(1-a_i)b_{ij}}{a_i}$. Marshallian demands, x_i and $z_i^k = z_k^i$, and indirect utility, v_i^k, of an individual i connected to individual k are given by:

$$x_i = \frac{I^i}{p_x}(1+2\theta)^{-1} \tag{96}$$

$$z_i^k = \frac{p_x}{p_z+p_y}\theta(x_i+x_k) = z_k^i = \frac{I^i+I^k}{p_z+p_y}\theta(1+2\theta)^{-1} \tag{97}$$

$$v_i^k = A(1+2\theta)^{-\mu_i}\left[\left(\frac{I^i}{p_x}\right)^{a_i}\left(\frac{I^i+I^k}{p_z+p_y}\theta\right)^{(1-a_i)}\right]^{\mu_i} \tag{98}$$

Internalizing equilibrium price formation — now allowing for any given number of individuals in the economy, n, where each of them mates one and only one individual:

$$\frac{p_z+p_y}{p_x} = 2\theta\frac{\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l} \tag{99}$$

$$x_i = \frac{w_x^i\sum_{l=1}^n w_z^l + w_z^i 2\theta\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l}(1+2\theta)^{-1} \tag{100}$$

$$z_i^k = z_k^i = \frac{(w_x^i+w_x^k)\sum_{l=1}^n w_z^l + (w_z^i+w_z^k)2\theta\sum_{l=1}^n w_x^l}{2\sum_{l=1}^n w_x^l}(1+2\theta)^{-1} \tag{101}$$

$$v_i^k = A(1+2\theta)^{-\mu_i} \tag{102}$$
$$\left\{\left(\frac{w_x^i\sum_{l=1}^n w_z^l + w_z^i 2\theta\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l}\right)^{a_i}\left[\frac{(w_x^i+w_x^k)\sum_{l=1}^n w_z^l + (w_z^i+w_z^k)2\theta\sum_{l=1}^n w_x^l}{2\sum_{l=1}^n w_x^l}\right]^{(1-a_i)}\right\}^{\mu_i}$$

$$\begin{aligned}\frac{p_z+t_i^j}{p_x} &= \frac{p_y-t_j^i}{p_x} = \frac{I_i}{I_i+I_j} = \frac{p_z+p_y}{p_x} \\ &= \frac{w_x^i\sum_{l=1}^n w_z^l + w_z^i 2\theta\sum_{l=1}^n w_x^l}{(w_x^i+w_x^k)\sum_{l=1}^n w_z^l + (w_z^i+w_z^k)2\theta\sum_{l=1}^n w_x^l}\frac{p_z+p_y}{p_x} \\ &= \frac{w_x^i\sum_{l=1}^n w_z^l + w_z^i 2\theta\sum_{l=1}^n w_x^l}{(w_x^i+w_x^k)\sum_{l=1}^n w_z^l + (w_z^i+w_z^k)2\theta\sum_{l=1}^n w_x^l}2\theta\frac{\sum_{l=1}^n w_x^l}{\sum_{l=1}^n w_z^l}\end{aligned} \tag{103}$$

Given the special form of the utility function — the linearity of demands for the private good, with fixed (for all i) marginal increment, in $\frac{I^i}{p_x}$ (and independence of mate's income — even if linearity with fixed marginal increment also in the latter would imply the same result) —, the relative full price level is independent of resource distribution. Also due to the uniformity of the direct utility functions, it is also independent of the particular mating arrangement that should come to develop in the economy.

Nevertheless, out of similar special cases, mating dynamics are expected to feedback to it.

6. ASSORTATIVE MATING AND TRANSFERABILITY

6.1. Introduction

In this section, we are going to suggest some of the expected mating arrangements in an economy where individual i ($i = 1, 2, \ldots, n$) possesses utility potential $v_i^k(I^i, I^k)$, where $I^{i(k)}$ is $i(k)$'s income, if paired with $k \neq i$, and the equilibrium devices involved in its determination. Obviously, $v_i^k(I^i, I^k)$ may represent an indirect utility function of individual i arising from a direct utility function exhibiting taste-for-unicity and an optimization involving shared-consumption — say, such as (96).

We will further assume that $v_i^k(I^i, I^k) = v_i(I^i, I^k)$, all k and i, that the same general indirect utility function form applies for all potential mates, only differing and increasing in their income level — i.e., $\frac{\partial \nu_i(I^i, I^k)}{\partial I^k} > 0$ for all i, k, and all the individuals I — with the first sub-index i left in the indirect utility function just to indicate the individual to which it belongs to. This is a simplifying assumption[18]: we might as well just require that any potential mate k is preference ordered — ranked — similarly by any i in the economy.

Everybody wants to mate with the highest income. He can just mate one individual... as also the second lowest income: mating types will constitute a relatively scarce resource, the usual setting under which pricing systems naturally develop. But for pricing to occur, one must be able to pay in some other resource — i.e., to trade. Given the context — $v_i(I^i, I^k)$ —, a plausible "numeraire" would then be income I^i[19]. Another, often encountered in the family economics literature, is utility — utility units — itself: utility is then invoked to be transferable between the couple.

If neither utility nor endowments (income...) are transferable — individuals "must" obtain utility according to $v_i(I^i, I^k)$, because $\frac{\partial \nu_i}{\partial I^k} > 0$ for all i, — more generally, because the ranking of potential mates in the economy is uniform —, we expect positive assortative mating in the econo-

[18] Form (96) obeys it due to the uniformity of direct preferences in the economy of the special case...

[19] We might as well consider one of the two endowments... We are assuming that any of them can.

my: higher income (more highly preferred as mate) individuals will cluster together starting at the highest level.

In other cases, different assignments may be generated. Some contexts have been thoroughly studied in the literature, namely, transferable utilities — see Legros and Newman (2002) for recent references[20]. However, not all cases; and when efficiency was analyzed, connection with the implicit supporting price system was missing. We therefore proceed to both.

6.2. Transferable Utilities

One can find in Becker (1973) a proof that, in the presence of transferable utilities, positive (negative) assortative mating is optimal in the sense that it maximizes the sum of individuals' utilities, positively dependent on the income of each of the individuals forming a pair, iff $\frac{\partial^2 \nu_i}{\partial I^i \partial I^k} > (<) 0$. The condition was later generalized to the requirement of super (sub) modularity — see for instance, Legros and Newman (2002) for a definition. In this sub-section, we provide an intuition (an alternative proof) for the result, when $\frac{\partial \nu_i}{\partial I^k} > 0$ for all i, after characterizing a first-order condition principle for efficient matching and relate it to the supporting (general equilibrium) pricing system. We further digress on the spontaneous mating arrangement arising when matching pairs are formed with individuals of distinct groups.

The marginal benefit obtained by individual i, with income I^i, by mating with individual k of income I^k, call it d_i^k, is the utility gain he obtains by mating with k instead of with the individual $k-1$ when potential mates are ordered by ascending order of income. I.e.:

$$d_i^k = v_i(I^i, I^k) - v_i(I^i, I^{k-1}) \tag{104}$$

In a decentralized economy, mating changes are expected to occur till equality of the marginal benefit of the match — the price (in utility units) that individuals would pay for the last match improvement - across the economy, i.e., for all the i's that mated; in the optimal assignment scheme:

$$d_i^{k^*} = v_i(I^i, I^{k^*}) - v_i(I^i, I^{k^*-1}) = p_{MF}, i = 1, 2, \ldots, n \tag{105}$$

Such rule would stem from first-order conditions for efficiency — characterized more generally in V.5 -, i.e., maximization of $\sum_{i=1}^{n} \nu_i(I^i, I^{k^*})$, which, at given individual income levels and in the presence utility transferability would appear as the natural maximand: the couple formed by i has joint utility maximized for $(n/2 - 1)$ given levels of sum of couple utilities we assign to other couples.

Let then the n individuals that are mated be ordered ascendingly according to their own income level, $i(k) = 1, 2, \ldots, n$. Then, i pays a "net"

[20] Also, Bulow and Levin (2006) — when both agent types to be matched maximize monetary objective functions, we may assume the hypothesis applies.

dowry[21] to k^*:

$$D_i^{k^*} = p_{MF}(r_{k^*} - r_i) \approx p_{MF}(k^* - i) \tag{106}$$

where $r_i(r_k)$[22] represents the rank order of individual $i(k)$ by individual $k(i)$'s preferences — and of all individuals above $k(i)$. I.e., i obtains "net-of-transfers" utility:

$$v_i^{k^*} = v_i(I^i, I^{k^*}) - D_i^{k^*} = v_i(I^i, I^{k^*}) - p_{MF}(k^* - i) \tag{107}$$

in the optimal match in which he is paired with k^*, the one chosen to operate utility transfers with. The equalization of the marginal benefit of mating with k to the ranking points price arises naturally from FOC of the discrete choice problem facing i of determining the k that maximizes $v_i^k = v_i(I^i, I^k) - p_{MF}(k - i)$ — once i, i cannot change… $v_i^{k^*} + v_{k^*}^i = v_i(I^i, I^{k^*}) + v_k(I^k, I^{i^*})$, all i, k^*, and therefore transfers are confined to each pair.

p_{MF} is the price of the income ranking points in the economy for matching purposes. Those points are attributed according to a classification that ranges from 1 to n[23], (i.e., even if there is income replication, in which case the rank of equally endowed individuals could be the mid-rank of the individuals in the category) where n is the number of individuals that were paired, discrete[24] and consecutive if all incomes differ. Such pricing scheme occurs, or is due, because unicity is required at the utility level — matching with j has the opportunity cost of not being available to match with somebody else.

In equilibrium, for individuals that were not mated by the matching market (that stayed outside the group of the n mated ones — i.e., such n

[21] See Botticini and Siow (2003) for a recent overview of other rationales for dowries and bequests.

[22] They can just slightly differ from $i(k)$ —at most, $i - r_i = 1, k - r_k = 1$ -, because one cannot mate with oneself…

[23] This preference ordering — quantifying quality — of the match with each individual, k, must be uniformly accepted and agreed upon in the economy — be independent of i - for the price system (competition or market for ranking points-discrete quantities, but nevertheless aggregatable quantities) to work. If not, and i_j is the preference ordering assessment of individual i by individual j in a scale of 1 (least preferred) to $n-1$ (most preferred) — so that i is endowed or rated with $\sum_{\substack{j=1 \\ j\neq i}}^{n} i_j$ points, uniquely appreciated by everybody —, one would speculate that an equilibrium condition could require $[v_i(i,k) - v_i(i,k-1)]/[\sum_{\substack{j=1 \\ j\neq k}}^{n} k_j - \sum_{\substack{j=1 \\ j\neq k-1}}^{n} (k-1)_j] = p$ to be constant in the optimal assignment, where k is i's pair — $v_i(i,k)$ i's utility when paired with k -, $(k-1)$ his next preference, and p the price of all ranking points in the market $-n\sum_{i=1}^{n-1} i = (n-1)n^2/2$ - with $D_i^k = p(\sum_{\substack{j=1 \\ j\neq k}}^{n} k_j - \sum_{\substack{j=1 \\ j\neq i}}^{n} i_j)$.

[24] The price will be that of a discrete ranking of potential partners, not of their income: what is as stake is a discrete location over a set of ordered alternatives. Of course, the income magnitude affects the equilibrium price but through its effect on utility levels.

is, or are, endogenous), it must be the case that for unmatched j's either:

$$d_j^{1^*} = v_j(I^j, I^{1^*}) - v_j(I^j, 0) < p_{MF}, j = n+1, n+2, \ldots \quad (108)$$

where I^{1^*} is the lowest income of the paired individuals. While the reverse is occurring — as in any market —, there is excess demand for matching and p_{MF} will be increasing while additional matches are being arranged, process that becomes complete only when equality holds - because of discreteness, till $d_i^{k+1^*} < p_{MF} \leq d_i^{k^*}$ - for all the (some...) n mated partners.

Or the closest mated income to the (an) excluded j, say $j+1^*$, is mated with someone — k^* - that would not change it for j. That is:

$$d_k^{j-1^*} = v_k(I^{k^*}, I^j) - v_k(I^{k^*}, I^{j-1^*}) < p_{MF}, j = n+1, n+2, \ldots \quad (109)$$

(108) would apply when lower incomes are not mated — arising with positive assortative mating; (109) when middle incomes are not mated, expected with negative assortative mating.

For the resulting arrangement to be optimal for individual i —for him to achieve the maximum and not the minimum utility with marginal benefit to price equalization -, one requires the marginal benefit to be decreasing in the match, i.e., $d_i^{k^*} - d_i^{k^*-1} = v_i(I^i, I^{k^*}) - v_i(I^i, I^{k^*-1}) - [v_i(I^i, I^{k-1^*}) - v_i(I^i, I^{k^*-2})] < 0$ — where k^*-2 is the next best match to (before income) k^*-1. This is satisfied if $\frac{\partial^2 \nu_i}{\partial I^{k^2}} < 0$[25] and existing income levels in the economy are equally spaced.

Now, for $d_i^{k^*}$ to be constant in the economy, I^i and the income of the pair, I^{k^*}, must change or relate according to (or close...) - differentiating (103):

$$\begin{aligned} & \frac{\partial d_i^k}{\partial I^i} dI^i + \frac{\partial d_i^k}{\partial I^{k^*}} dI^{k^*} + \frac{\partial d_i^k}{\partial I^{k^*}} dI^{k^*-1} \\ = & \left[\frac{\partial \nu_i(I^i, I^{k^*})}{\partial I^i} - \frac{\partial \nu_i(I^i, I^{k^*-1})}{\partial I^i} \right] dI^i + \frac{\partial \nu_i(I^i, I^{k^*})}{\partial I^k} dI^{k^*} \\ - & \frac{\partial \nu_i(I^i, I^{k^*-1})}{\partial I^k} dI^{k^*-1} = 0 \end{aligned} \quad (110)$$

[25]As in conventional continuous optimization, non-convexities — e.g., increasing returns to scale — may generate equilibrium failure, as well as validity of interior FOC of the efficient allocation solution.

Assume that income levels are equally or uniformly spaced in the economy so that $dI^{k^*} = dI^{k^*-1}$. Then:

$$\left[\frac{\partial\nu_i(I^i, I^{k^*})}{\partial I^i} - \frac{\partial\nu_i(I^i, I^{k^*-1})}{\partial I^i}\right] dI^i = -\left[\frac{\partial\nu_i(I^i, I^{k^*})}{\partial I^{k^*}} - \frac{\partial\nu_i(I^i, I^{k^*-1})}{\partial I^{k^*-1}}\right] dI^{k^*} \tag{111}$$

Approximately, $\frac{\partial\nu_i(I^i,I^{k^*})}{\partial I^i} - \frac{\partial\nu_i(I^i,I^{k^*-1})}{\partial I^i} \approx (I^{k^*} - I^{k^*-1})\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^i\partial I^k}$ and $[\frac{\partial\nu_i(I^i,I^{k^*})}{\partial I^{k^*}} - \frac{\partial\nu_i(I^i,I^{k^*-1})}{\partial I^{k^*-1}}] \approx -(I^{k^*} - I^{k^*-1})\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^{k^2}}$. Then we expect the assignment in the economy to exhibit:

$$\frac{\partial^2\nu_i(I^i, I^{k^*})}{\partial I^i\partial I^k} dI^i = -\frac{\partial^2\nu_i(I^i, I^{k^*})}{\partial I^{k^2}} dI^{k^*} \tag{112}$$

If $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^{k^2}} < 0$ (required by SOC for maximum benefit), then $\frac{dI^{k^*}}{dI^i} > 0$ and we register positive assortative mating — as income rises, so does that of the partner — iff $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^i\partial I^k} > 0$. $\frac{dI^{k^*}}{dI^i} < 0$ and we register negative assortative mating — as income rises, that of the partner tends to decrease — iff $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^i\partial I^k} < 0$.

Similar conclusions would be obtained if we reasoned with the marginal loss from accepting k^* instead of the next upper income, $l_i^{k^*} = v_i(I^i, I^{k^*+1}) - v_i(I^i, I^{k^*}) = \text{constant}, i = 1, 2, \ldots, n$. Provided $\frac{\partial^2\nu_i}{\partial I^{k^2}} < 0$ and income is evenly spaced, $l_i^{k^*} < d_i^{k^*}$.

If $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^{k^2}} > 0$, marginal benefit equalization leads to minimum individual (and, thus, aggregate) utility; such minimization would be consistent with assignments such that $\frac{dI^{k^*}}{dI^i} < 0$, i.e., negative (positive) assortative mating, iff $\frac{\partial^2\nu_i}{\partial I^i\partial I^k} > (<)0$. But, when SOC fail, the marginal equalization principle — and the law of one price — fails: demands for match ranking points are no longer negatively sloped. Then, one would expect that if $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^i\partial I^k} > 0$, a match with simultaneously high income of partners generates a higher utility surplus, transferable within the couple, and there would be positive assortative mating; with $\frac{\partial^2\nu_i(I^i,I^{k^*})}{\partial I^i\partial I^k} < 0$, a match with dissimilar income levels would; i.e., we always (still) expect - because utility is transferable - the equilibrium assignment to be the optimal aggregate one. But the failure of the market match price equalization would confer bargaining power within some range to individuals within each pair — and lead to multiple possible arrangements of effective transfers occurring within the couple, eventually colliding with the optimality conditions generating the indirect utility functions...

Admit that mating can only occur between an individual of group M (males, $1, 2, \ldots, n_A$) and another of group F (females, $n_A{+}1, n_A{+}2, \ldots, n$).

One could think that different prices could be formed for rankings of each group, say p_M for ranking points of males — equalized to the marginal benefit that individuals of group F are deriving from mating with those of group M — and p_F for those of females — the marginal benefit that males are deriving from mating with females[26]. An individual of group $M(i = 1, 2, \ldots, \min(n_M, n_F)$, ordered ascendingly by income on group M, where $\min(n_M, n_F)$ are individuals that end-up effectively mated) would pay to an individual of group $F(k = 1, 2, \ldots, \min(n_M, n_F)$, ordered ascendingly by income on group F) a net transfer $D_i^{k^*} = p_F^{k^*} - p_M^i, i = 1, 2, \ldots, \min(n_M, n_F); k = 1, 2, \ldots, \min(n_M, n_F)$; consistently, an individual of group $F(k = 1, 2, \ldots, \min(n_M, n_F))$ would pay to an individual of group $M(i = 1, 2, \ldots, \min(n_M, n_F))$ a net transfer $D_k^{i^*} = p_M^{i^*} - p_F^k, k = 1, 2, \ldots, \min(n_M, n_F); i = 1, 2, \ldots, \min(n_M, n_F)$; the individual of each group would equalize his marginal benefit to the price of the ranking points of the other group. Yet, equilibrium would not yet be defined, once it requires additionally an overall appraisal of the two groups relative income availability. Moreover, interpersonal comparison with the own group rankings end up by being made indirectly, which is not accounted for by that pricing system.

One would therefore speculate that the previous — uniform pricing — rule still applies, with marginal benefit and ranking order of individuals — unique and uniquely priced — being calculated as if one could also mate with people of the own group; the equilibrium price of ranking points now adjusts till k^* belongs to the opposite group. Or that, under group-specific rankings, $p_M(p_M \sum_{\substack{i=1 \\ i \in F}}^{\min(n_M, n_F)} k^*)$ and $p_F(p_F \sum_{\substack{i=1 \\ i \in M}}^{\min(n_M, n_F)} k^*)$ will approximate: the marginal benefit of a mate in the economy — the price of ranking points for matching purposes — would attempt to equalize.

Under unbalanced groups, the last rule may, again not be sufficient. If there is:

- positive assortative mating: prices should guarantee that $d_j^{1^*} = v_j(I^j, I^{1^*}) - v_j(I^j, 0) < p_F$ if $n_A > n - n_A$ and only $n - n_A M$'s are mated; to $d_{j'}^{1^*} = v_{j'}(I^{j'}, I^{1^*}) - v_{j'}(I^{j'}, 0) < p_M$ if $n_A < n - n_A$ and only $n_A F$'s are mated — with 1^* the lowest income mated of the other group — for individuals j (of M), j' (of F) not mated (that preferred not to match in the optimal assignment) of each group. Given the positive sorting, low income levels are expected to be excluded, and the highest excluded income qualifies the relevant marginal unmated individual, j or j'. And due to the evolution of marginal benefit, the price approximation rule may be sufficient.

[26]As equalization of marginal benefit for each group equalizes, cross-derivative correspondence with the sign of sorting is still be valid.

• negative assortative mating: prices will go up till — guarantee that $-d_i^{k^*} = v_j(I^{i^*}, I^j) - v_j(I^{i^*}, I^{k^*}) < p_M$ if $n_A > n - n_A$ and only $n - n_A M$'s are mated; to $d_i^{k^*} = v_j(I^{i^*}, I^{j'}) - v_{j'}(I^{i^*}, I^{k^*}) < p_F$ if $n_A < n - n_A$ and only $n_A F$'s are mated — with i^* the individual mated with next lowest income relative to the excluded (not mated) individuals j (of M), j' (of F) of each group. Given the negative assorting, middle income levels are expected to be excluded, and the lowest excluded income qualifies the relevant marginal unmated individual, j or j', j or j'.

With positive assortative mating, the effective transfer between the pairs in a couple tends to 0. Yet, the ranking points price system must be at least latent — insuring (provided SOC hold) equalization of the marginal benefit across the economy and not other (non-optimal in the presence of utility transferability) mating rule. With negative assortative mating, non-negligible transfers effectively occur between pairs.

6.3. Transferable Income

If utility is not transferable across individuals but income is, one could advance that the marginal benefit equated across individuals would be measured in income terms, i.e., ${d'}_i^k$ such that[27]:

$$\begin{aligned} & v_i(I^i - {D'}_i^{k^*-1} - {d'}_i^k, I^{k^*} + {D'}_i^{k^*-1} + {d'}_i^k) \\ = \; & v_i(I^i - {D'}_i^{k^*-1}, I^{k^*-1} + {D'}_i^{k^*-1}) \end{aligned} \tag{113}$$

that is:

$$\begin{aligned} & v_i[I^i - (k^* - i)p_{MF}, I^{k^*} + (k^* - i)p_{MF}] \\ = \; & v_i[I^i - (k^* - 1 - i)p_{MF}, I^{k^*-1} + (k^* - 1 - i)p_{MF}] \end{aligned} \tag{114}$$

Individual i chooses k maximizing $v_i[I^i - (k - i)p_{MF}, I^k + (k - i)p_{MF}]$, which would generate FOC implying that the difference between the left and right hand-sides of (112) — the marginal net-of-cost benefit — approaches zero.

p_{MF} is now a price measured in income units and ${D'}_i^{k^*}$ deducted to the individual i's own resources. It reflects the fact that a couple's budget constraints or resources can be pooled, and it incorporates a measure of the strength of the individual in the household allocation decision.

[27]These are also the expected market features if both utility and income are transferable, provided that $v_i(I^i, I^k)$ is quasi-concave in the two arguments: i chooses k^* by making the derivative of $v_i(I^i, I^k)$ with respect to k^* — the difference between the left and right-hand side terms of each of the expressions — equal to zero.

Using Taylor's expansion to the first order we can (grossly...) approximate:

$$d'^{k^*}_i \approx \frac{\nu_i(I^i, I^{k^*}) - \nu_i(I^i, I^{k^*-1})}{\frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^i} - \frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^k}} \tag{115}$$

$$\approx \frac{\nu_i(I^i, I^{k^*})}{\frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^i} - \frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^k}} - \frac{\nu_i(I^i, I^{k^*-1})}{\frac{\partial \nu_i(I^i,I^{k^*-1})}{\partial I^i} - \frac{\partial \nu_i(I^i,I^{k^*-1})}{\partial I^k}} = p_{MF}$$

Then, for adequate conclusions on mating one would advance that if the function[28]

$$v'_i(I^i, I^k) = \frac{\nu_i(I^i, I^{k^*})}{\frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^i} - \frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^k}} \tag{116}$$

that evaluates i's utility in terms of income units, positively related to I^k iff $\frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^k} > v'_i(I^i, I^k)[\frac{\partial^2 \nu_i(I^i,I^{k^*})}{\partial I^i \partial I^k} - \frac{\partial^2 \nu_i(I^i,I^{k^*})}{\partial {I^k}^2}]$ (provided that $\frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^i} > \frac{\partial \nu_i(I^i,I^{k^*})}{\partial I^k}$) - is concave in I^k, $\frac{dI^*}{dI^i} > (<) 0$ and we register positive (negative) assortative mating iff $\frac{\partial^2 \nu'_i(I^i,I^{k^*})}{\partial I^i \partial I^k} > (<) 0$.

6.4. Absence of Transferability

If neither utility nor income are transferable, we may speculate that willingness to form a pair will still be ruled by the previous mechanism — a matching market. Yet, the equilibrium is going to press the actual transfer between individuals of each couple to zero — not to equalization of marginal benefit, but of its product by the couple ratings differential to zero, i.e.:

$$D_i^{k^*} = [v_i(I^i, I^{k^*}) - v_i(I^i, I^{k^*-1})](k^* - i) = 0, i = 1, 2, \ldots, n \tag{117}$$

Positive assortative mating is then always expected — the absolute value of $(k^* - i)$ being minimized:

$$k^* \approx i, i = 1, 2, \ldots, n^{29} \tag{118}$$

[28] $v'_i(I^i, I^k)$ can be seen as inversely related to "boldness" — see Aumann and Kurz (1977) -, the semi-elasticity of the utility with respect to the argument; here, the denominator is deducted from the compensating effect through the partner's income.

[29] If $v_i(i, k)$ is i's utility when paired with k and i_j is the preference ordering assessment of individual i by individual j — in a scale of 1 (least preferred) to $n-1$ (most preferred) —, one can adventure a simple algorithm that under non-transferable utilities would join i and k such that $\sum_{\substack{j=1 \\ j\neq i}}^n i_j \approx \sum_{\substack{j=1 \\ j\neq k}}^n k_j, i, k = 1, 2, \ldots, n$ - that is, minimizing the average absolute distance between the rankings in each duo (provided all i's are considered acceptable to k and vice-versa — i.e., with unacceptability to j of partner i,

as forwarded in the beginning of the section.

Here, n would include all individuals. If there are two groups, then rankings (here exogenous and fixed...) go from 1 to n for the largest group, from the difference in elements between the two groups plus 1 to n for the smallest.

Without transferability of any sort, such equilibrium is efficient as well.

6.5. The Efficient Allocation

Some final appraisal on mating efficiency can be forwarded. Firstly, none of the conditions qualifies social efficiency: this requires a social welfare function and also some redistribution possibilities over utility, its arguments or through match dictation... With transferable utility, a Benthamite — maximizing sum of individuals' utilities[30] — optimization criterion does not guarantee a social optimum for all possible welfare functions either: the transfer dictated by the latter, not by the Benthamite one, would also have to effectively take place afterwards...

Also, never do we expect to approach a pure Benthamite result: the transfers occur only between members of a couple. On the one hand, the maximization rule of the sum of utilities invoked before applies only to the transferable utility case, and on the other, refers to the sum of "indirect" utilities...

An efficient allocation with monogamous matching and transferable utilities — through mating but not other — can be linked to a problem of type (8), for monogamous utility functions, with (8) replaced by $\max_{x_i, z_i^j, x_j, z_j^l, k, l, j} U^i(x_i, z_i^k, y_i^k) + U^k(x_k, z_k^i, y_k^i)$ and (8a) by $U^j(x_j, z_j^l, y_j^l) + U^l(x_l, z_l^j, y_l^j) \geq \overline{U}^j + \overline{U}^l = \overline{U'}^{jl}, j \neq i, k, lj, l = 1, 2, \ldots, n$ (and j with l only); or in a more complex formulation, with (8) replaced by $\max_{x_i, z_i^j, x_j, z_j^l, k, l, j, \overline{U}^k, \overline{U}^j, \overline{U}^l} U^i(x_i, z_i^k, y_i^k) + \overline{U}^k$ and added of $\overline{U}^j + \overline{U}^l \geq \overline{U'}^{jl}$, $jl = 1, 2, \ldots, n/2 - 1$. Or yet, (8) is replaced by $\max_{x_i, z_i^j, (w_i - w_k), x_j, z_j^l, (w_j - w_l), k, l, j} U^i(x_i, z_i^k, y_i^k) + (w_i - w_k)$ and (8a) by $U^j(x_j, z_j^l, y_j^l) + (w_j - w_l) \geq \overline{U}^j j \neq i, j = 1, 2, \ldots, n$: transfers are adjustable to provide optimal partnership well-being.

Without transferability, (8) is just replaced by $\max_{x_i, z_i^j, x_j, z_j^l, k, l, j} U^i(x_i, z_i^k, y_i^k)$.

With transferable endowments to a mate — but not other nor utility —, given that the shared good must be consumed at the same lev-

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the algorithm should be so constrained, and possibly allow i_j to be 0 for such cases, and j choose i_j in the scale of 1 to the number of acceptable choices to him/her out of the total individuals minus 1 — of the maximum individuals in each group, n, if one cannot match with the same group, for which i_j would start at the difference plus 1.) See Gale and Shapley (1962) and Roth (1984) on optimal assignment.

[30]Which, in any case, it is not our general implicit criterion — only for matching purposes...

el for both partners but not the other, we hypothesise that (8) becomes $\max_{x_i,z_i^j,(w_i-w_k),x_j,z_j^l,(w_j-w_l),k,l,j} U^i(x_i+w_i-w_k,z_i^k,y_i^k)$ and (8a) $U^j(x_j+w_j-w_l,z_j^l,y_j^l) \geq \overline{U}^j j \neq i, j = 1,2,\ldots,n$: income transferability between partners allows any allocations $x_i + x_j = x_i^* + x_j^*$ where the latter are the solution found for two partners i and j — then, transfers are adjustable to provide optimal partnership well-being. (Of course, for appropriate $\overline{U}^j$'s, the problem applying to the no transferability case, $\max_{x_i,z_i^l,x_j,z_j^l,k,l,j} U^i(x_i,z_i^k,y_i^k)$, generates the same solution as that of the current paragraph.)

Transferability of both endowments and utility between individuals in a pair would imply replacing (8) by $\max_{x_i,z_i^j,x_j,z_j^l,k,l,j} U^i(x_i,z_i^k,y_i^k)+U^k(x_k,z_k^i,y_k^i)$ and (8a) by $U^j(x_j,z_j^l,y_j^l)+U^l(x_l,z_l^j,y_l^j) \geq \overline{U'}^{jl}, j \neq i,k,lj,l = 1,2,\ldots,n$ (and j with l only): no definition of individual utility levels would be supplied…

As noted in section 3, the Samuelson condition is expected to hold in any of the efficient allocations.

6.6. Cobb-Douglas Preferences: An Example

We can apply the previous rules to our utility function[31]. Using (95), $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$ — there will be no assortative "matching" — nor positive, nor negative; but the qualification relies here on the interpretation of the cross effect only on the level of z_i^k (per couple). To conclude about couple formation, one must rely on the indirect utility function properties:

From (98), $\frac{\partial \nu_i}{\partial I^k} = \mu_i(1-a_i)v_i\frac{1}{(I^i+I^k)} > 0$; $\frac{\partial^2 \nu_i}{\partial I^{k2}} = \mu_i(1-a_i)[\mu_i(1-a_i)-1]v_i\frac{1}{(I^i+I^k)^2}$. As $\frac{\partial^2 \nu_i}{\partial I^i \partial I^k} = \mu_i(1-a_i)v_i\frac{\mu_i a_i I^k+(\mu_i-1)I^i}{(I^i+I^k)^2 I^i} > 0$ iff $I^k > \frac{1-\mu_i}{\mu_i a_i}I^i$, positive assortative "mating" is expected if the direct utility function exhibits constant or increasing returns to scale — and utility is transferable across individuals.

An individual of any type will prefer to mate an individual with higher income — a higher v_i^k. If $\mu_i \geq 1$ (IRS or CRS), there will be correspondence; then linkages will sort themselves by decreasing income levels. With DRS, if in the economy, for any $i,k,I^i > \frac{\mu_i a_i}{1-\mu_i}I^k$, negative assorting can occur — with strongly decreasing returns to scale and a low relative preference for the individual private good; if the reverse happens, we still observe positive assorting in couple formation.

In sum, with non-decreasing returns to scale, "doubly-good" marriages will be popular — but these not necessarily longer or with more children (not involving higher z_i^k's) than just a couple's pooled income implies — because $\frac{\partial^2 z_i^k}{\partial I^i \partial I^k} = 0$…

[31]See Becker (1973), p. 826 and 841, and Lam (1988).

If utility is not transferable but income is, the mating qualification would rely on the cross effects over the function:

$$v_i'(I^i, I^k) = \frac{\nu_i(I^i, I^k}{\frac{\partial \nu_i(I^i,I^k)}{\partial I^i} - \frac{\partial \nu_i(I^i,I^k)}{\partial I^k}} = \mu_i^{-1} a_i^{-1} I^i \tag{119}$$

$\frac{\partial \nu_i'}{\partial I^i} = \mu_i^{-1} a_i^{-1} > 0 (\frac{\partial \nu_i'}{\partial I^i} = 0)$ and $\frac{\partial^2 \nu_i'}{\partial I^i \partial I^k} = 0$: with non-transferable utility and transferable income, no assortative mating is expected.

6.7. Final Discussion

Congestion of linkages — say, a fixed number of linkages — would also generate a ranking market. Say r links are supported by each individual and indirect utilities are of the form $v_i(I^i, I^{k_1}, I^{k_2}, \ldots, I^{k_r})$ and utility is transferable; it is possible that, with I_{k_i*} ordered ascendingly, that the equilibrium will imply that for all individuals (and one relevant group) $v_i(I^i, I^{k_1*}, I^{k_2*}, \ldots, I^{k_r*}) - v_i(I^i, I^{k_1*-1}, I^{k_2*}, \ldots, I^{k_{r-1}*}, I^{k_r}) = p =$ constant — i solves $\max_{k_1,k_2,\ldots,k_{r-1},k_r} v_i(I^i, I^{k_1}, I^{k^2}, \ldots, I^{k_r}) + pri - pk_1 - pk_2 - \cdots - pk_r$ —, where I^{k_1*-1} is the income of the highest income lower to I^{k_1}.

Illustrating special arrangements, some of social others of engineering interest:

Case A. Group Formation. $1 \longleftrightarrow 2 \qquad 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0; z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

Links between 1 (2) and 3 are too expensive. Such case may arise either due to 3's utility function valuing less communication (z's and y's) than the others; or by either 1 and 2's (or all...) utility functions embedding strong substitutability between links with different individuals (between z_i^j and $z_i^{j'}$; z_i^j and $y_i^{j'}$; and between y_i^j and $y_i^{j'}$; y_i^j and $z_i^{j'}$), but not with the same (i.e., not between z_i^j and y_i^j; nor $z_i^{j'}$ and $y_i^{j'}$).

Case B. Transit Sequence. $1 \longleftrightarrow 2 \longleftrightarrow 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0$.

If utility is related to distance — and 1 and 3 are more distant than 2 is to either 1 or 3 — a transit sequence appears.

Case C. One-Way Transit Sequence. $1 \longrightarrow 2 \longrightarrow 3$

$z_1^3 = y_1^3 = 0$ and $z_3^1 = y_3^1 = 0; z_2^1 = y_2^1 = 0; z_3^2 = y_3^2 = 0$

This case may also suggest a multiple layer hierarchy.

1

Case D. Hierarchic Sequence. $\nearrow \quad \nwarrow$

2 3

$z_1^2 = y_1^2 = 0; z_1^3 = y_1^3 = 0; z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

Attention of 1 seems more important than that of all other individuals. Notice that it may mean that equilibrium specific-transfers obtained from 1 are relatively higher in equilibrium.

Case E. Emission Sequence.

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ 2 & & 3 \end{array}$$

$z_2^1 = y_2 61 = 0$; $z_3^1 = y_3^1 = 0$; $z_2^3 = y_2^3 = 0$ and $z_3^2 = y_3^2 = 0$

1 may be an advertising point. Or, in a hierarchic chain, it has a leading role with respect to the purchase of (decisions over) z.

Case F. One-Way Circular Sequence.

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \nwarrow \\ 2 & \longrightarrow & 3 \end{array}$$

$z_2^1 = y_2^1 = 0$; $z_2^3 = y_2^3 = 0$; $z_1^3 = y_1^3 = 0$

7. PUBLIC GOOD VS. SHARED GOOD

In this section, we inspect the case where the externality is extended to more than one consumer, even if to a fixed number: if the number is not fixed, we would fall under a typical club good case. There will be an efficient allocation but the market may no longer insure its attainment...

Assume then that each z is in fact consumed by the whole economy. $z_i^j = y_j^i = y_j$. Then each z_i^j — as y_j — is replicated among the n consumers. Let us then admit it is unique or uniform. i's utility takes the form

$$U^i(x_i, z_i, y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \tag{120}$$

We will denote it by $U^i(x_i, z_i, y_{-i})$. I obtains utility from the private good, x_i, from its own purchases of the public good, z_i, and from the purchases other consumers make, y_j, so that:

$$z_j = y_j, j = 1, 2, \ldots, n \tag{121}$$

Of course, each z_i is then a conventional public good — we have n different public goods in the economy. A special case where a common (unique) public good is formed arises for $U^i(x_i, z_i, y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) = U^i(x_i, z_i + y_1 + y_2 + \cdots + y_{i-1} + y_{i+1} + \cdots + y_n)$ with (119) holding.

Assume (120) — with (121). An efficient allocation will be obtained from the problem:

$$\max_{x_i, z_i, y_i, x_j, z_j, y_j} U^i(x_i, z_i, y_{-i}) \tag{122}$$

s.t.:

$$U^j(x_j, z_j, y_{-j}) \geq \overline{U}^j, j \neq i, j = 1, 2, \ldots, n \tag{123}$$

$$z_i = y_i, i = 1, 2, \ldots, n \tag{124}$$

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} W_x^i \tag{125}$$

$$\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} W_z^i \tag{126}$$

In lagrangean form, embedding (124):

$$\max_{x_i, z_i^j, x_j, z_j^i, \lambda_j, \mu_x, \mu_z} U^i(x_i, z_i, z_{-i}) + \sum_{\substack{j \neq i \\ j=1}}^{n} \lambda_j [\overline{U}^j - U^j(x_j, z_j, z_{-j})] \tag{127}$$

$$+ \mu_x \left(\sum_{i=1}^{n} W_x^i - \sum_{i=1}^{n} x_i \right) + \mu_z \left(\sum_{i=1}^{n} W_z^i - \sum_{i=1}^{n} z_i \right)$$

Interior FOC require:

$$U_x^i - \mu_x = 0 (1 \text{ equation}) \tag{128}$$

$$-\lambda_j U_x^j - \mu_x = 0, j \neq i, j = 1, 2, \ldots, n (n - 1 \text{ eqs.}) \tag{129}$$

$$U_{z_k}^i - \sum_{\substack{j \neq i \\ j=1}}^{n} \lambda_j U_{z_k}^j - \mu_z = 0, k = 1, 2, \ldots, n (n \text{ equation}) \tag{130}$$

(128) and (129) imply (16) that still holds

$$\lambda_j = -\frac{U_x^i}{U_x^j}, j \neq i, j = 1, 2, \ldots, n \tag{131}$$

Replacing (131) in (130), and equating the two (and (128)) we obtain the familiar Samuelson condition(s):

$$\sum_{j=1}^{n} \frac{U_{z_i}^j}{U_x^j} (= \frac{\mu_z}{U_x^i}) = \frac{\mu_z}{\mu_x}, i = 1, 2, \ldots, n (n \text{ equations}) \tag{132}$$

Let us consider a price-cum-transfer system analogous to that of the call to decentralize that efficient solution. Each consumer i pays p_z for z_i and p_y per unit of y_j, i.e., by $z_j, j \neq i$; he pays $t_i^j, j \neq i$, to each of the other $n-1$ individuals for accepting his choice of z_i and receives t_j^i from each for per unit he accepts of their choice of z_j. A typical budget constraint is then:

$$p_x x_i + \left(p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j \right) z_i + \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i) z_j = p_x W_x^i + p_z' W_z^i \tag{133}$$

The lagrangean will take the form:

$$\max_{x_i,z_i,\mu} U^i(x_i,z_i,z_{-i}) \tag{134}$$
$$+\mu\left[p_xW_x^i+p_z'W_z^i-p_xx_i-\left(p_z+\sum_{\substack{j\neq i\\ j=1}}^{n}t_i^j\right)z_i-\sum_{\substack{j\neq i\\ j=1}}^{n}(p_y-t_j^i)z_j\right]$$

and FOC for $i=1,2,\dots,n$:

$$U_x^i-\mu p_x=0 \tag{135}$$
$$U_{z_i}^i-\mu\left(p_z+\sum_{\substack{j\neq i\\ j=1}}^{n}t_i^j\right)=0 \tag{136}$$
$$U_{z_j}^i-\mu(p_y-t_j^i)=0, j\neq i, j=1,2,\dots,n \tag{137}$$

with the budget constraint. Then:

$$\frac{U_{z_i}^i}{U_x^i}=\frac{p_z+\sum_{\substack{j\neq i\\ j=1}}^{n}t_i^j}{p_x}, i=1,2,\dots,n(1 \text{ eq. for each } i) \tag{138}$$

and

$$\frac{U_{z_j}^i}{U_x^i}=\frac{p_y-t_j^i}{p_x}, j\neq i, j=1,2,\dots,n(n-1 \text{ eqs. for each } i) \tag{139}$$

Equilibrium requires additionally:

$$p_z'=p_z+(n-1)p_y \tag{140}$$
$$\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}W_x^i \tag{141}$$
$$\sum_{i=1}^{n}z_i=\sum_{i=1}^{n}W_z^i \tag{142}$$

A full price system can be derived: (138) and (139) and individual budget constraints generate $n\times(n+1)$ equations that add to (140)-(142): $n\times(n+1)+3$ equations with (the sum of budget constraints making) one of the last three redundant. We must generate $2n$ individual consumptions, and a vector price $(\frac{p_z}{p_x},\frac{p_y}{p_x},\frac{p_z'}{p_x},\frac{t_1^2}{p_x},\dots,\frac{t_1^n}{p_x},\dots,\frac{t_n^2}{p_x},\dots,\frac{t_n^{n-1}}{p_x})$ — with $n\times(n-1)+3$ elements, i.e., $n(n+1)+3$ unknowns (also the nx_i's and nz_i's). Again if we fix, $\frac{p_z}{p_x}$ or $\frac{p_y}{p_x}$, a determined solution is obtained for all relative prices (in terms of good x).

But if under one-to-one communication, replication of individuals of each type may insure competitive link-specific transfer price formation — we know who to charge what (even if we fix one price) given the actual transfer —, now, such possibility may no longer exist — and the natural spontaneity of the equilibrium breaks down...

I.e., competitive decentralization requires — apart from absence of transaction costs — a smaller number of individuals types than the total number of individuals in the economy — and responsibility for each part of, or the common purchase to be assigned to someone — some type — in particular. With some agent heterogeneity, the final cost shares will be in line with marginal utilities. But — as is well-known — perfect information and type discrimination must then be ensured.

If i cannot veto z_j for $j \neq i$, — he does not directly obey (137) and, therefore, (139) — but authorities guarantee the (adequate) price $(p_z + \sum_{\substack{j \neq i \\ j=1}}^{n} t_i^j$ for the unit of z_i and collect as a lump-sum $Z_i = \sum_{\substack{j \neq i \\ j=1}}^{n} (p_y - t_j^i)z_j$ from (each) i, the efficient allocation is also ensured ((139) becomes redundant) — but then not entirely through the market price system.

8. SUMMARY AND CONCLUSIONS

General equilibrium of a pure exchange economy was proven to be able to generate efficient allocations in economies where share goods are present; under special arrangements, uniqueness is also guaranteed. Efficient allocations require the Samuelson (public good) rule with respect to the ratio of utilities — whether or not sharing takes the form of an externality. Optimal pricing involves common reception and emission prices — adding up to a uniquely determined quantity — along with link-specific transfers from consumers who value a specific "call" more than its charged price. End-specific roles — for adequate general price allocation — must also be pre-ordained — achieved with a (much) milder version of (than) the Arrow's dictator.

With multiple sharing by more than two individuals — because either the good is shared by more than one individual or because there are similar links between different pairs —, some indeterminacy may arise with respect to the distribution of the general aggregate unit cost. Of course, heterogeneity requires more complex identification.

CES utility functions generate interesting environments. With transferable utility, positive assortative mating is likely to arise with linear homogeneity or higher — and negative with strong DRS and/or low relative preferences for joint-consumption. Cobb-Douglas technologies, generating linear Engel curves, suggest no quantity assorting of household good demand.

Utility functions implying monogamy allowed us to study mating arrangements more profoundly. Definition of the marginal benefit of a match — and price of ranking points — was forwarded, and mechanics of an adequate (dowry) price system for an endogenous matching market explained; with transferable utility, the requirement of equalization of marginal benefit of a match across individuals provides the direction of assortative mating. If utility is not transferable but income — qualifying assorting — is, then it is the income value of the marginal benefit that is expected to equalize in the economy; this suggests the importance of the function given by the ratio of utility over the difference between the marginal utility relative to own income minus the marginal with respect to the partner's in determining the outcome of decentralized assorting.

Fruitful extensions are expected in family economic modelling and estimation, both in the static as in the intertemporal domain, with household decisions also covering labor market participation and supply, allowing for joint family investment — and taxation —, encompassing both single and multi-element unit as special cases, possibly assuming single and married, male and female (with or without children —, parameter preference differentiation.

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