The Term Structure of Interest Rates in a New Keynesian Model with Time-Varying Macro Volatility*

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We show that the New Keynesian sticky price model with a cost-push shock and time-varying volatilities of driving forces can reproduce the behavior of the U.S. yield curve in the post-World War II period. Furthermore, we examine how the yield data affects the estimation of time-varying volatilities. We find that if we omit the cost-push shock, we can get very different estimates of the inflation target volatility depending on whether or not we use yield data in addition to macroeconomic data. Therefore, the cost-push shock is crucial for a good prediction of the yield curve. We finally show that the slope of the yield curve depends negatively on both the volatility of the inflation target and the volatility of the cost-push shock.

Key Words: Term structure of interest rates; New Keynesian model; Time-varying volatility.
JEL Classification Numbers: E43, E44, E47.

1. INTRODUCTION

Recent literature has analyzed dynamic term structure models which are derived from macroeconomic models (Wu, 2005; Hoerdahl and Tristani, 2004). The idea is to combine aggregate demand and supply equations with a monetary policy rule to determine the short term interest rate. Absence of arbitrage is then used to derive the implied model of long-term yields. Within models of this kind, time-varying volatility of macro variables is a candidate explanation for the violation of the expectation hypothesis (Campbell and Shiller, 1991; Singleton, 2006). Besides, it is a stylized fact that volatilities of U.S. macro-aggregates have changed during the post-

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WWII period (Justiniano and Primiceri, 2006; Stock and Watson, 2005; McConnell and Perez-Quiros, 2000).

This raises some interesting questions. How are movements in the yield curve linked to variations in volatilities of macro variables? For example, in a period in which the volatility of the central bank's inflation target is high, investors might demand a different term premium than in a period in which the volatility of the central bank's inflation target is low. How well does a model implied yield curve, as described, fit observed yields? Note that the cited macroeconomic models assumed homoskedastic shocks and did not address this question. How does yield data affect the estimates of volatilities of stochastic driving forces in a macroeconomic model? If time-varying volatilities of macro variables are to produce a model-implied yield curve which fits the observed yield curve well, then the fit of the model implied yield curve should not be significantly better if, in the estimation, yield data is used in addition to macro data.

We address these questions within the framework of a simple New Keynesian sticky price model. New Keynesian models have become very popular for monetary policy analysis. They are consistent with optimizing behavior by private agents while being simple from a conceptual point of view (Walsh, 2003; Woodford, 2003; Clarida *et al.* , 1999; McCallum and Nelson, 1999). We estimate the model by maximum likelihood. We use a rolling window of U.S. post-war data in order to estimate time-varying volatilities of shocks. In one estimation, we use only macro data whereas in another estimation, we use yield data in addition to macro data. Volatilities estimated with macro and yield data are defined as implied volatilities whereas volatilities estimated solely with macro data are called actual volatilities.

We find that the fit of the model implied yield curve improves substantially (by a factor of 1.4) when the model is estimated with yield data. It appears that a simple New Keynesian model with time-varying actual volatilities (i.e. estimated only with macro data) is not rich enough to capture movements in yields. Since we find that the volatility of the inflation target is much lower when estimated with yield data than when estimated without, we deduce that the New Phillips curve in the model should be modified. Indeed, adding a cost-push shock to the New Phillips curve improves the fit of the model implied yield curve by a factor of 2.2. Furthermore, the model-implied yield curve turns out to improve only marginally if we use yield data in the estimation. Besides, the gap between implied volatility and actual volatility of the inflation target shock has become smaller. We explain this finding as follows. In the model without the cost-push shock, the actual volatility of the inflation target is driven by observed inflation. Therefore, this model does not have the required degree of freedom for a well fitting yield curve. In the model with the cost-push shock, however, it is the volatility of the cost-push shock which describes the volatility of observed inflation. Thus, the volatility of the inflation target is left over to produce a well fitting yield curve. Therefore, we conclude that a New Keynesian sticky price model with a cost-push shock and time-varying actual volatilities (i.e. estimated without yield data) succeeds in reproducing movements in the term structure of interest rates. Besides, we show that the slope of the model implied yield curve is negatively related to the volatility of the inflation target and to that of the cost-push shock.

In related work (Doh, 2007), the yield curve is used to infer the central bank's inflation target in a New Keynesian sticky price model. The novelty of our paper is the focus on time-varying volatilities and on the fit of the model implied yield curve. Another difference is the estimation method. In the cited work, the entire model has been estimated using Bayesian methods.

This paper is organized as follows. Section 2 reviews the Keynesian sticky price model. Section 3 shows how the yield curve is obtained. Section 4 presents the estimation method and Section 5 the results. Section 6 concludes the paper.

2. A BENCHMARK NEW KEYNESIAN MODEL

We assume that the economy is of the sticky-price type (Walsh, 2003; Woodford, 2003; Clarida *et al.* , 1999; McCallum and Nelson, 1999; Gali, 2003). In this section, we briefly review the main equations of the model.

2.1. Households

The representative household chooses a consumption path $(C_j)_{j\geq t}$ and a labor path $(N_j)_{j\geq t}$ in order to maximize expected life-time utility given by

$$\mathbb{E}_t \sum_{j=t}^{\infty} \beta^j U(C_j, N_j).$$

C_t is a composite consumption good which consists of differentiated products $C_t(i)$ produced by monopolistic competitive firms. It is defined by

$$C_t = \left[\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1.$$

The parameter ϵ governs the price elasticity of demand for the individual goods. The optimal allocation of expenditures is given by

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t \tag{1}$$

where $P_t(i)$ is the price of good i and P_t is a consumption goods price index defined by

$$P_t = \left[\int_0^1 P_t(i)^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}}.$$

The budget constraint is given by

$$P_t C_t + \sum_{\tau=1}^{J} b(\tau, t) B_{\tau,t} \leq \sum_{\tau=0}^{J-1} b(\tau, t) B_{\tau+1,t-1} + W_t N_t - T_t,$$

where $b(\tau, t)$ is the price of a bond at time t which expires in τ months, $B_{\tau,t}$ bond holding, W_t the nominal wage, and T_t taxes. The bonds are assumed to be zero coupon bonds which pay one unit of cash at expiration date $t + \tau$. Therefore, $b(0, t) = 1$ for all t.

We assume a utility function which is additively separable in consumption and leisure

$$U(C_t, N_t) = \frac{(C_t/A_t)^{1-\nu}}{1-\nu} - \frac{N_t^{1+\varphi}}{1+\varphi},$$

where A_t is an exogenous labor-augmenting technological change (Solow, 1956) and ν and φ are positive real numbers. The dependency of the utility on the ratio C_t/A_t, and not on C_t, is not standard. We see it as a simple way to introduce a feature which has a similar effect as external habit formation (Abel, 1990; Campbell and Cochrane, 1999). That is to say it makes the coefficient of relative risk aversion time-varying. This coefficient is given by

$$-\frac{C_t U_{cc,t}}{U_{c,t}} = \nu A_t.$$

Economically, it means that agents dislike situations in which their productivity is high whereas their consumption is relatively low due to other shocks.

2.2. Firms

On the production side, there is a continuum of monopolistically competitive firms, indexed by i. Each firm produces a differentiated good with technology

$$Y_t(i) = A_t N(i),$$

where A_t is the productivity level and $N(i)$ the firm's labor force. The abstraction of capital follows literature (McCallum and Nelson, 1999) which shows that, at least for the United States, there is little relationship between the capital stock and output at business-cycle frequencies. The firms face Calvo-type price stickiness (Calvo, 1983). This is to say that, in each period, some firms are not able to adjust their price. The probability that a firm can adjust the price is given by $1-\theta$. Hence, the parameter θ is a measure of the degree of nominal rigidity; a larger θ implies that fewer firms adjust each period and that the expected time between price changes is longer. The implied average price duration is $1/(1-\theta)$. If a firm adjusts the price, it does so to maximize the expected discounted value of current and future profits. Mathematically, this can be expressed as

$$\max_{P_t^*(i)} \sum_{j=t}^{\infty} \mathbb{E}_t \left\{ \theta^j Q_{t,j} \left(P_t^*(i) Y_{j|t}(i) - W_j N_j(i) \right) \right\},$$

where $Q_{t,j}$ is the stochastic discount factor of the firms and $Y_{j|t}$ is the production of a firm in period j if it could adjust its price in period t. $Y_{j|t}$ has to be equal to the optimal allocation $C_j(i)$ of households given by (1).

The firm sells its goods to private households and to the government.[1] Therefore, total demand is given by $C_t(i)+G_t(i)$, where G_t are government purchases. Following extant literature (Gali, 2003), we assume that the government consumes a fraction ς_t of each $Y_t(i)$.

2.3. Stochastic Driving Forces

The growth rates of technology and of government expenditures follow an autoregressive process of order two. They are described by

$$\begin{aligned} \Delta a_t &= \mu_a + \rho_{a,1}\Delta a_{t-1} + \rho_{a,2}\Delta a_{t-2} + \sigma_a \varepsilon_t^a \\ \Delta g_t &= \mu_g + \rho_{g,1}\Delta g_{t-1} + \rho_{g,2}\Delta g_{t-2} + \sigma_g \varepsilon_t^g, \end{aligned}$$

where Δ is the first difference operator, $a_t = \log A_t$ and $g_t = -\log(1-\varsigma_t)$. We introduced the second lag in order to accommodate potentially higher persistence in monthly data as compared to quarterly data.

2.4. Flexible Price Equilibrium

In the flexible price equilibrium ($\theta = 0$) firms put a constant common markup on their prices given by $\frac{\epsilon}{\epsilon-1}$. This implies that real marginal costs are constant and equal to minus $\log\frac{\epsilon}{\epsilon-1} = \mu$ for all firms. This leads to the key equation for finding the flexible price equilibrium:

$$-\mu = w_t - p_t - a_t - s,$$

where w_t is the logarithm of the nominal wage, $p_t = \log P_t$ and s a wage subsidy. Combining this equation with the log-version of the optimality

[1] Government consumption can be viewed as the sum of usual government expenditures plus net exports (Chari *et al.* , 2007).

condition $-\frac{\partial U}{\partial N_t} = \frac{\partial U}{\partial C_t}\frac{W_t}{P_t}$,

$$\varphi n_t = w_t - p_t - \nu(c_t - a_t),$$

and with $y_t = a_t + n_t$, provides the solution for y_t. Using this solution in the log-version of the Euler equation,

$$y_t = -\frac{1}{\nu}(r_t - \mathbb{E}_t[\pi_{t+1}] - \rho) + \mathbb{E}_t[y_{t+1} - \Delta g_{t+1} - \Delta a_{t+1}],$$

gives the solution for the flexible price anticipated real interest rate. We denote by $\overline{y}_t$ and $\overline{rr}_t$ the flexible price log-output and the flexible price real interest rate, respectively. In equilibrium, we have

$$\begin{aligned}
\overline{y}_t &= \gamma + \psi_a a_t + \psi_g g_t \\
\overline{rr}_t &= r_t - \mathbb{E}_t[\pi_{t+1}] \\
&= \rho + \nu(\psi_a - 1)[\mu_a + \rho_{a,1}\Delta a_t + \rho_{a,2}\Delta a_{t-1}] + \\
&\quad +\nu(\psi_g - 1)[\mu_g + \rho_{g,1}\Delta g_t + \rho_{g,2}\Delta g_{t-2}],
\end{aligned}$$

where $\gamma = \frac{s-\mu}{\nu+\rho}$, $\psi_a = \frac{1+\nu+\varphi}{\nu+\varphi}$ and $\psi_g = \frac{\nu}{\nu+\varphi}$.[2] We assume that $\gamma = 0$. This means that the equilibrium allocation under flexible prices coincides with the allocation under flexible prices and perfect competition. We recall that this is attained by an employment subsidy s which offsets the distortions associated with monopolistic competition.

2.5. Monetary Policy

We complete the model with a monetary policy rule of the Taylor type. Concretely, the central bank sets r_t according to:

$$\begin{aligned}
r_t &= \rho_r r_{t-1} + (1-\rho_r)(\tilde{\rho} + \pi_t^* + \phi_\pi(\mathbb{E}_t[\pi_{t+1}] - \pi_t^*) + \phi_y x_t) + \sigma_r \varepsilon_t^r \\
\varepsilon_t^r &\sim iidN(0,1),
\end{aligned}$$

where $\tilde{\rho} = \rho + \nu((\psi_a - 1)\mu_a + (\psi_g - 1)\mu_g)$, $r_t = \log\{R_t\}$ with R_t denoting the nominal gross interest rate and π_t^* the central bank's targeted inflation rate (as given by the logarithm of a price ratio).[3] We assume that the target π_t^* has a random walk representation:

$$\pi_t^* = \pi_{t-1}^* + \sigma_\pi \varepsilon_t^{\pi^*} \quad , \quad \varepsilon_t^{\pi^*} \sim iidN(0,1).$$

[2] See Gali (2003) for a derivation with different assumptions on the stochastic driving forces.

[3] Taking the logarithm of the price ratio as the inflation rate (target) has an important advantage over taking $(P_t - P_{t-1})/P_{t-1}$: Whereas the latter lies in $(-1, \infty)$, the former can take on any value in $\mathbb{R}$. Therefore, the log-inflation rate is consistent with normally distributed disturbances. The same applies for $r_t = \log R_t = \log\{1 + i_t^{FF}\}$, where i_t^{FF} stands for the observed federal funds rate.

2.6. Sticky Price Equilibrium

For the staggered price setting case ($\theta > 0$), the approximate equilibrium conditions are given by

$$\begin{aligned} y_t &= c_t + g_t && (2)\\ x_t &= -\nu^{-1}(r_t - \mathbb{E}_t[\pi_{t+1}] - \overline{rr}_t) + \mathbb{E}_t[x_{t+1}] && (3)\\ \pi_t &= \beta\mathbb{E}_t[\pi_{t+1}] + \kappa x_t, && (4)\end{aligned}$$

where $x_t = y_t - \overline{y}_t$ is the output gap, $\pi_t = \log\{P_t/P_{t-1}\}$ the inflation rate and r_t the central bank rate. Further, $\kappa = (1-\theta)(1-\beta\theta)(\nu+\varphi)/\theta$. The monetary policy rule closes the model. The solution is more difficult to obtain than for the flexible price economy. We therefore use a standard algorithm to derive it (Sims, 2002). This algorithm yields the rational expectation solution of the model in the form

$$\xi_t = C + \Theta_0\xi_{t-1} + \Theta_1\varepsilon_t \tag{5}$$

where C is a vector of constants and Θ_0 and Θ_1 are deterministic matrices built up from the structural parameters of the model. ξ_t is given by

$$\xi_t = \big(y_t\ \overline{y}_t\ x_t\ c_t\ g_t\ g_{t-1}\ g_{t-2}\ a_t\ a_{t-1}\ a_{t-2}\ r_t\ \overline{rr}_t\ \pi_t\ \pi_t^*\ \mathbb{E}_t\pi_{t+1}\ \mathbb{E}_t x_{t+1}\ m_t\ u_t\big).$$

The structural shocks ε_t^a, ε_t^g, $\varepsilon_t^{\pi^*}$ and ε_t^r are stacked in ε_t.

3. LINKING THE ECONOMY AND THE TERM STRUCTURE

We obtain the term structure of interest rates from the Euler equation. The Euler equation is given by

$$b(\tau,t) = \beta\mathbb{E}_t\left[\frac{U_{c,t+1}/P_{t+1}}{U_{c,t}/P_t}b(\tau-1,t+1)\right] = \mathbb{E}_t[M_{t+1}b(\tau-1,t+1)]. \tag{6}$$

The stochastic discount factor is well visible in this equation. We define it by

$$M_{t+1} = \beta\frac{U_{c,t+1}/P_{t+1}}{U_{c,t}/P_t}.$$

Taking logs and using the explicit expression of the utility function, we get

$$m_{t+1} = \log M_{t+1} = \log\beta - \nu((c_{t+1}-a_{t+1}) - (c_t - a_t)) - \pi_{t+1}.$$

Hence, a bond which pays one unit when consumption is higher than productivity (or when prices are high), costs less than a bond which pays one unit when consumption is low (or when prices are low). We put m_t into ξ_t in (5). Therefore, the closed form linearized solution of the stochastic discount factor is given by the respective line in (5). We write it as

$$m_{t+1} = \mu_m + \theta'_{0,m}\xi_t + \theta'_{1,m}\varepsilon_{t+1}. \tag{7}$$

It is well known that if the term structure is inferred by using (5) in the linearized version of (6), then we would get

$$\log b(\tau,t) - \log b(\tau-1,t+1) = \mathbb{E}_t[m_{t+1}], \tag{8}$$

for all τ. This means that ex-ante returns would be equal across bonds with different maturities. In other words, no term premium exists and the model-implied term structure would be flat. Therefore, this approach is not qualified for the question at hand. One solution consists of doing a second order approximation of the model (Lombardo and Sutherland, 2005; Schmitt-Grohé and Uribe, 2004). However, this would substantially increase the computational burden as it requires a particle filter to infer the likelihood function (Fernandez-Villaverde and Rubio-Ramirez, 2006). Instead, we use a well-known approach (Jermann, 1998). It is based on the assumption that M_{t+1} and $b(\tau,t)$ have a log-normal distribution. This assumption implies that the Euler equation (6) can then be written as (first, using the moment generating function, then taking logarithms)

$$\begin{aligned} \log b(\tau,t) &= \mathbb{E}_t[m_{t+1} + \log b(\tau-1,t+1)] \\ &\quad + \frac{1}{2} Var_t[m_{t+1} + \log b(\tau-1,t+1)]. \end{aligned} \tag{9}$$

Thereby, m_{t+1} is assumed to be given by (7). (9) differs from (8) by a variance-covariance term. This term captures the term premium. It expresses that $b(\tau,t)$ depends positively on the conditional variances and the covariance. This introduces heterogeneity in ex-ante return. Evaluating (9) at $\tau=1$ gives the continuously compounded one period bond return, $r_{t,1}$ (considering that $b(0,t)=1$):

$$\begin{aligned} -r_{t,1} = \log(b(1,t)) &= \mathbb{E}_t[m_{t+1}] + (1/2)Var_t[m_{t+1}] \\ &= \mu_m + \theta'_{0,m}\xi_t + (1/2)\theta'_{1,m}\theta_{1,m}. \end{aligned} \tag{10}$$

In order to determine other bond prices, equation (9) has to be solved using the boundary conditions $b(0,t)=1$ for all t. The unique solution to this difference equation is of exponential affine form (Vasicek, 1977; Duffie *et al.* , 2002) and given by

$$b(\tau,t) = \exp\left(A(\tau) + B(\tau)'\xi_t\right) \tag{11}$$

for some functions A and B to be determined. Plugging (11) into the difference equation (9) verifies that it is indeed a solution. With (7) and (11), (9) can be written as

$$\begin{aligned}
\log b(\tau, t) &= \mu_m + \theta'_{0,m}\xi_t + A(\tau-1) + B(\tau-1)'[C + \Theta_0\xi_t] \\
&\quad +2^{-1}Var_t[\theta'_{1,m}\varepsilon_{t+1} + B(\tau-1)'\Theta_1\varepsilon_{t+1}] \\
&= \mu_m + A(\tau-1) + 2^{-1}\theta'_{1,m}\theta_{1,m} + B(\tau-1)'(\Theta_1\theta_{1,m} + C) \\
&\quad +2^{-1}B(\tau-1)'\Theta_1\Theta'_1 B(\tau-1) + (\theta'_{0,m} + B(\tau-1)'\Theta_0)\xi_t.
\end{aligned}$$

Since, by (11), $\log b(\tau, t)$ has to be equal to $A(\tau) + B(\tau)'\xi_t$, A and B have to solve the following system of difference equations:

$$\begin{aligned}
A(\tau) &= A(\tau-1) + \mu_m + 2^{-1}\theta'_{1,m}\theta_{1,m} + B(\tau-1)'(\Theta_1\theta_{1,m} + C) \\
&+ \ 2^{-1}B(\tau-1)'\Theta_1\Theta'_1 B(\tau-1) \\
&= A(\tau-1) + A(1) + B(\tau-1)'(\Theta_1\theta_{1,m} + C) \\
&+ \ 2^{-1}B(\tau-1)'\Theta_1\Theta'_1 B(\tau-1) \\
B(\tau) &= \theta_{0,m} + \Theta'_0 B(\tau-1) \\
&= B(1) + \Theta'_0 B(\tau-1).
\end{aligned}$$

We use the boundary conditions $b(0, t) = 1 \ \forall t$ to solve these equations recursively. The recursion starts with

$$\begin{aligned}
A(1) &= \mu_m + 2^{-1}\theta'_{1,m}\theta_{1,m} \\
B(1) &= \theta_{0,m}.
\end{aligned}$$

Given A and B, it is now straightforward to obtain the model implied term structure. The continuously compounded yield to maturity $t + \tau$ is defined by

$$\begin{aligned}
r_{\tau,t} &= (1/\tau)(\log(b(0, t+\tau)) - \log(b(\tau, t))) \\
&= -\log(b(\tau, t))/\tau \\
&= -\frac{1}{\tau}A(\tau) - \frac{1}{\tau}B(\tau)'\xi_t. \qquad (12)
\end{aligned}$$

$r_{\tau,t}$ as a function of τ is the model implied term structure. Note that (12) is a multi-factor term structure model where the dynamics of the factors are obtained from a macro-economic model. This kind of term structure models has been discussed (Wu, 2005; Hoerdahl and Tristani, 2004).

3.1. Market Prices of Risk and Time-Varying Term Premium

The expected one-period excess return of holding a bond with maturity $t+\tau$ is defined as

$$\mathbb{E}_t[\log b(\tau-1,t+1)] - \log b(\tau,t) - r_{1,t}.$$

Plugging (11) into this equation, gives

$$\begin{aligned}\mathbb{E}_t[\log b(\tau-1,t+1)] - \log b(\tau,t) - r_{1,t} &= -B(\tau-1)'\Theta_1\theta_{1,m} \\ &\quad -\frac{1}{2}B(\tau-1)'\Theta_1\Theta_1 B(\tau-1)'.\end{aligned} \tag{13}$$

$B(\tau-1)'\Theta_1$ is the coefficient vector which links $\log b(\tau-1,t+1)$ to ε_{t+1} which is the source of uncertainty between periods t and $t+1$. It has been argued that the second term on the right-hand-side is negligible (Wu, 2005). This means that $-B(\tau-1)'\Theta_1\theta_{1,m}$ can be interpreted as the compensation, or term premium, of holding $B(\tau-1)'\Theta_1$ units of the risks ε_{t+1}. Hence, $-\theta_{1,m}$ is the vector with the market prices of the risks ε_{t+1}. Evaluating (13) successively at (τ,t), $(\tau-1,t+1)$,..., summing these evaluations and taking conditional expectations at t leads to

$$r_{\tau,t} = \frac{1}{\tau}\sum_{k=0}^{\tau-1}\mathbb{E}_t[r_{1,t+k}] + p_\tau, \tag{14}$$

where p_τ is the term premium built from the right-hand side of (13). This equation states that the long term yield equals expected future short term rates plus a premium. This is known as the expectation hypothesis. It has been found that (14) is not supported by the data, which has been attributed to time-varying term premiums (Campbell and Shiller, 1991; Campbell, 1995; Rudebusch and Wu, 2004; Bacchetta *et al.* , 2009). In the present framework, a time-varying term premium may result from either time-varying structural parameters or from time-varying volatilities. This is the motivation for using the term structure in order to estimate volatilities.

4. ESTIMATION

4.1. Data and Calibration

We use U.S. macro and treasury bond data, measured at monthly frequency in order to estimate the volatilities and the parameters of the stochastic driving forces. The sample period is January 1970 to September 2008. We take personal consumption expenditures (PCE), deflated by the corresponding price index (PCEPI), to measure C_t. We further use the price index PCEPI to compute inflation. We proxy Y_t with a monthly total production index. For the instrument r_t of the central bank we take the federal funds rate. The bond yields are from Fama CRSP discount bond yield files and Datastream. We use all maturities of one to 13 months and further maturities of 24, 25, 36, 37, 48, 49, 60, 61, 84, 85 and 120 months. We obtain the actual volatilities by estimating the model a second time with macro data only, that is to say with C_t, Y_t, the inflation rate and r_t. We take extant estimates (Doh, 2007) to calibrate the structural parameters. Since these estimations were done for quarterly data, we make suitable adaptations (see Table 1). We then estimate the processes of the stochastic shocks in the model. We do another estimation with $\nu = \varphi = 1$ and otherwise identical parameters. $\nu = 1$ means that utility is logarithmic in consumption and $\varphi = 1$ implies a unit wage elasticity of labor supply (Gali, 2003).

TABLE 1.

Calibration of Sticky Price Model

Parameter	Calibration 1	Calibration 2
β	0.999	0.999
ν	1.9455	1
φ	3.3223	1
ρ	0.001	0.001
ρ_r	$0.418^{1/3}$	$0.418^{1/3}$
ϕ_π	2.034	2.034
ϕ_y	0.782	0.782
θ	$0.421^{1/3}$	$0.421^{1/3}$

4.2. Econometric Method and Implementation

The system to be estimated is given by

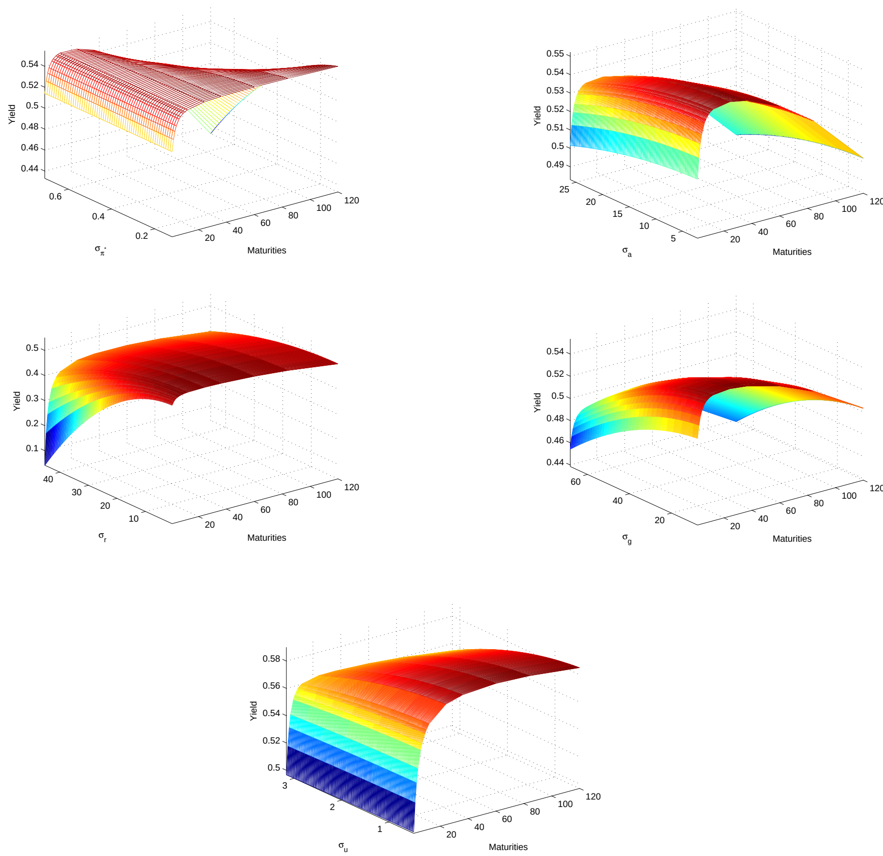
$$\begin{aligned} d_t &= A + B\xi_t + R\, w_t && (15) \\ \xi_{t+1} &= C + \Theta_0 \xi_t + \Theta_1 \varepsilon_{t+1}, && (16) \end{aligned}$$

where d_t contains either the observed macro variables and the yields,

$$d_t = \left(y_t \; c_t \; r_t \; \pi_t \; r_{1,t} \; r_{2,t} \; \ldots \; r_{120,t} \right),$$

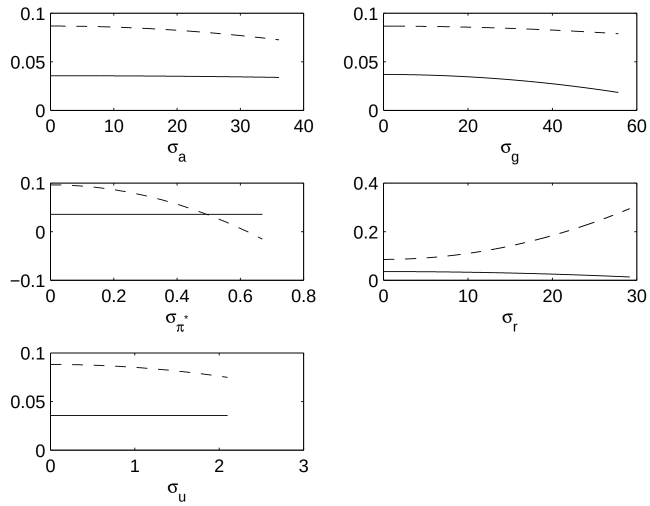
or only the macro variables. w_t is a standard multivariate normal measurement error which we assume to be independent of ε_{t+1}. We set R such that there are no measurement errors on the macro variables and, if it applies, a measurement error of 0.0005 on the yields (0.0005 corresponds to 8-10% of the average yield). R can be interpreted as a weighting matrix. It allows us to regulate the importance of yield data in the estimation. The estimation method is maximum likelihood. We obtain the likelihood function with the Kalman filter. Since direct maximization turned out to have convergence problems, we used an EM algorithm (Durbin and Koopman, 2001). This algorithm has two steps, i.e. an expectation evaluation step and a maximization step. We briefly summarize these steps, adapted to the present problem. Thereby, ψ denotes the parameter vector to be estimated and d the vector of all data vectors $d_1, \ldots, d_T$ (idem for ξ).

FIG. 1. Yield Curve Dependency on Volatilities



STEP 1
Let $\tilde{\psi}$ be a trial parameter vector. For this trial value, run the Kalman Smoother to get the smoothed states.

FIG. 2. Level (solid) and Slope (dashed) of Model Implied Yield Curve



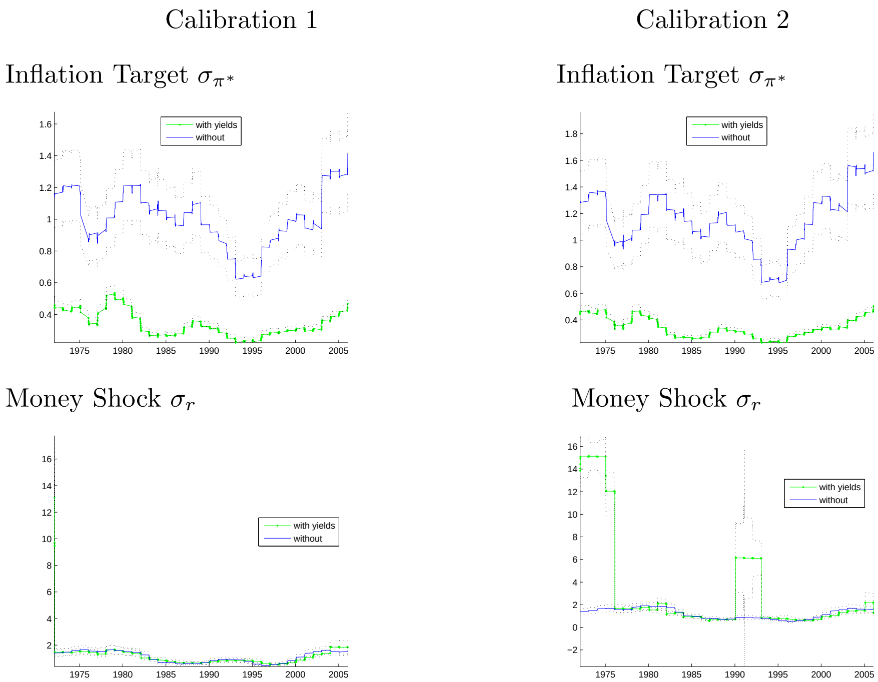
For each macroeconomic state, its arithmetic average over the entire sample period (1970-2008) is used. Level and slope of the model implied yield curve at these averaged states are then computed.

STEP 2
By Bayes rule, the marginal density p *of the data given the parameters* ψ *satisfies*

$$\begin{aligned}\log p(d|\psi) &= \log p(\xi^s, d|\psi) - \log p(\xi^s|d,\psi) \\ &= \tilde{\mathbb{E}}[\log p(\xi^s, d|\psi)] - \tilde{\mathbb{E}}[\log p(\xi^s|d,\psi)],\end{aligned}$$

where $\tilde{\mathbb{E}}$ *is the expectation operator with respect to density* $p(\xi^s|d,\tilde{\psi})$ *and* ξ^s *a vector with four linearly independent smoothed states (for example* a*,* g*,* π^* *and* r*) as obtained from step 1. Since the economy is driven by four shocks and since there are no predetermined variables, using more than four states would lead to singular covariance matrices. The gradient vector at*

FIG. 3. Implied (with term structure) and Actual (without term structure) Volatilities



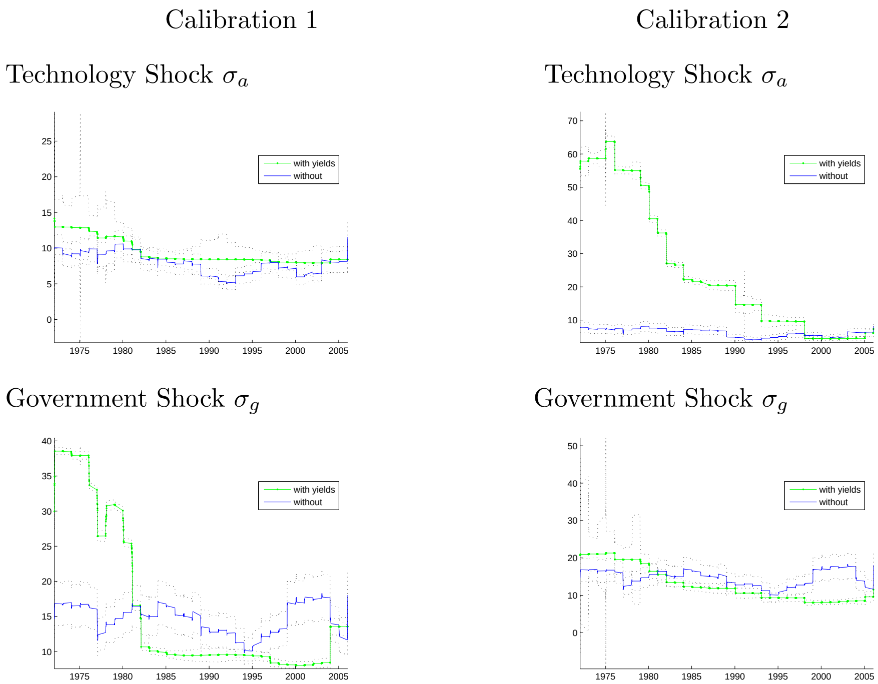
the trial value is given by

$$\begin{aligned}
\frac{\partial \log p(d|\psi)}{\partial \psi}|_{\psi=\tilde{\psi}} &= \tilde{\mathbb{E}}\left[\frac{\partial \log p(\xi, d|\psi)}{\partial \psi}\right]|_{\psi=\tilde{\psi}} \\
&= -\frac{1}{2}\frac{\partial}{\partial \psi}\sum_{t=1}^{T}(\log|RR'| + \log|(\Theta_1^s)(\Theta_1^s)'| \\
&+ (\xi_{t|T}^s - C^s - \Theta_0^s \xi_{t-1|T}^s)'((\Theta_1^s)(\Theta_1^s)')^{-1}(\xi_{t|T}^s - C^s - \Theta_0^s \xi_{t-1|T}^s) \\
&\quad (d_t - A' - B'\xi_{t|T})'(RR')^{-1}(d_t - A' - B'\xi_{t|T}))|_{\psi=\tilde{\psi}}.
\end{aligned}$$

Thereby, $(\Theta_1^s)(\Theta_1^s)'$ is diagonal with elements σ_a^2, σ_g^2 etc. This suggests to take as a new trial value in step 1 the ψ for which

$$\tilde{\mathbb{E}}\left[\frac{\partial \log p(\xi, d|\psi)}{\partial \psi}\right] = 0.$$

FIG. 4. Implied (with term structure) and Actual (without term structure) Volatilities



The first order conditions for σ_a (σ_g, σ_π and σ_r are analogous) is

$$T\frac{1}{\sigma_a} - \sum_{t=1}^{T} \frac{(\Delta a_t - \rho_1 \Delta a_{t-1} - \rho_2 \Delta_{t-2})^2}{\sigma_a^3} \tag{17}$$
$$+ \sum_{t=0}^{T} \frac{\partial}{\partial \sigma_a}(d_t - A' - B'\xi_{t|T})'(RR')^{-1}(d_t - A' - B'\xi_{t|T})) = 0.$$

RR' is diagonal and can be viewed as a weighing of the yield part: if its diagonal elements are very big, then σ_a is as estimated without yield data.

We assume that the initial states, in step 1, have a diffuse prior density (Durbin and Koopman, 2001). First, we estimate the system for the entire sample period. Then, we take a rolling window of 60 months and estimate only the volatilities for each window. This is to say that we do not redraw the states. Instead, we use the filtered states and estimated autocorrelations of the shocks from the first estimation. Besides the implied volatilities (as obtained from (17)), we also infer the actual volatilities for each window by doing the estimation without the yield curve (actual volatility is given by (17) without the third summand).

5. RESULTS

We first consider how the model implied yield curve depends on the volatilities (Figures 1 and 2). For this, we evaluate the model at data averages and at the estimates obtained from the estimation over the entire period with yield-data. We let the volatility of interest vary from its smallest to its biggest estimate. A first finding is that the volatility of the inflation target negatively influences the slope of the model implied yield curve but not its level (the *slope* being defined as the difference between the yield with the longest maturity and the short term yield, the *level* as the unweighted average of the yields over all maturities). The volatility of the short term monetary shock, σ_r, has an effect on both level and slope. The volatilities of the technology shock and of the government shock have a minor effect on level and slopes.

Figures 3 and 4 show actual (estimated without yield data) and implied (estimated with yield data) volatilities with 95% confidence intervals (dotted lines) for both calibrations. The implied volatilities of technology and government shocks differ substantially between the two calibrations. A decay over time is the only thing they appear to have in common. We find that implied volatilities of the monetary policy related shocks do not change a great deal across calibration. Implied volatility of the inflation target, however, is significantly lower than actual volatility. We were wondering whether introducing a first order autocorrelated cost-push shock would narrow this gap.

A cost-push shock modifies the equilibrium condition for inflation (4) to

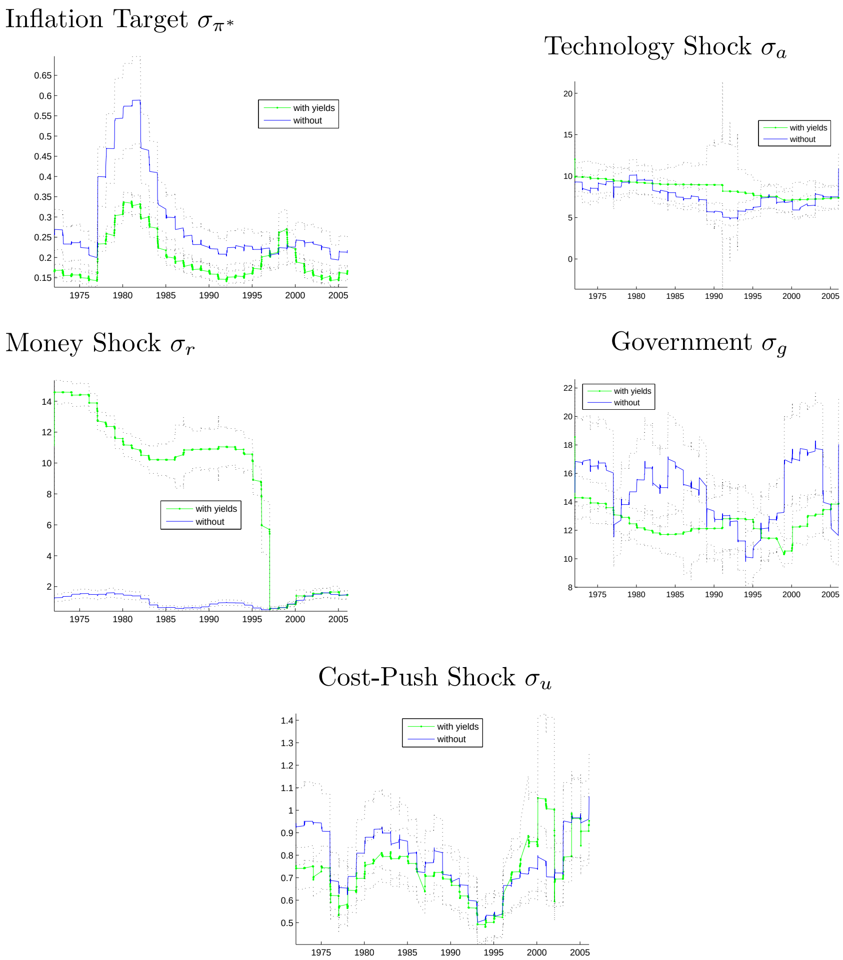
$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1}] + \kappa x_t + u_t,$$

where $u_t = \rho_u u_{t-1} + \sigma_u \varepsilon_t^u$ *is the cost-push shock (Gali, 2003).*

Not surprisingly, it does indeed, as Figure 5 shows (the calibration of the parameters in the model with the cost-push shock is the same as calibration one). Figure 2 indicates that the volatility of the cost-push shock negatively affects the slope of the yield curve.

The cost-push shock volatility resembles the volatility of central bank's inflation target in Figure 3. Furthermore, both these volatilities resemble the volatility of the first difference of observed inflation (Figures 6 and 7). This suggests that in the model without the cost-push shock, the volatility of the inflation target is driven by observed inflation. By contrast, in the model with cost-push shock it is the volatility of the cost-push shock which describes the volatility of observed inflation. This causes variations in the

FIG. 5. Volatilities with cost-push shock model (calibration 1)



volatility of the inflation target to be more tightly linked to variations in yield data. Note that the slope of the model implied yield curve is more sensitive to the volatility of the inflation target than it is to the volatility of the cost-push shock (Figure 2). The fit of the model implied yield curve of the cost-push shock economy should therefore be better than the fit of the model implied yield curve of the economy without cost-push shock. This conjecture is supported by the following assessment based on Table 2. Table 2 indicates how well the model implied yield curve fits the observed

FIG. 6. $\Delta\pi_t$

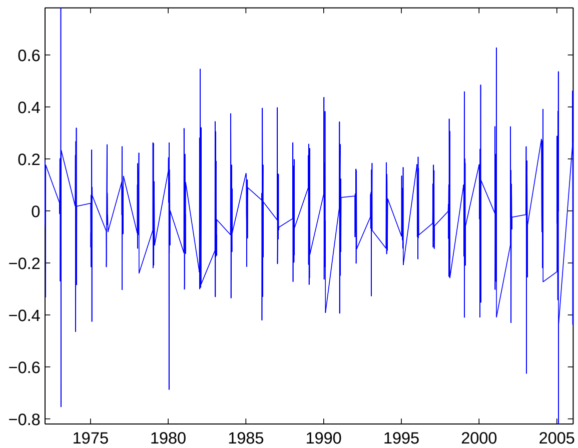
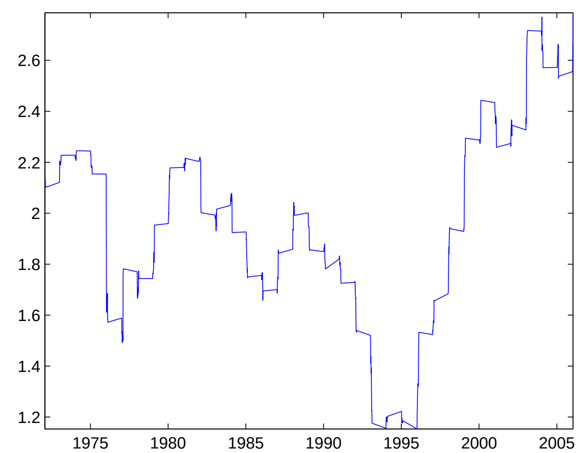


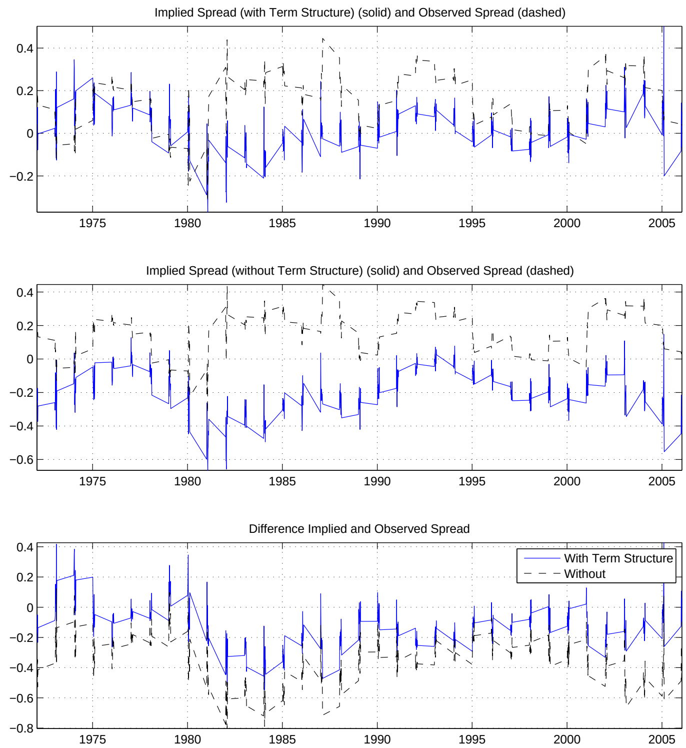
FIG. 7. $\sigma(\Delta\pi_t)$



yield curve. The stated numbers are

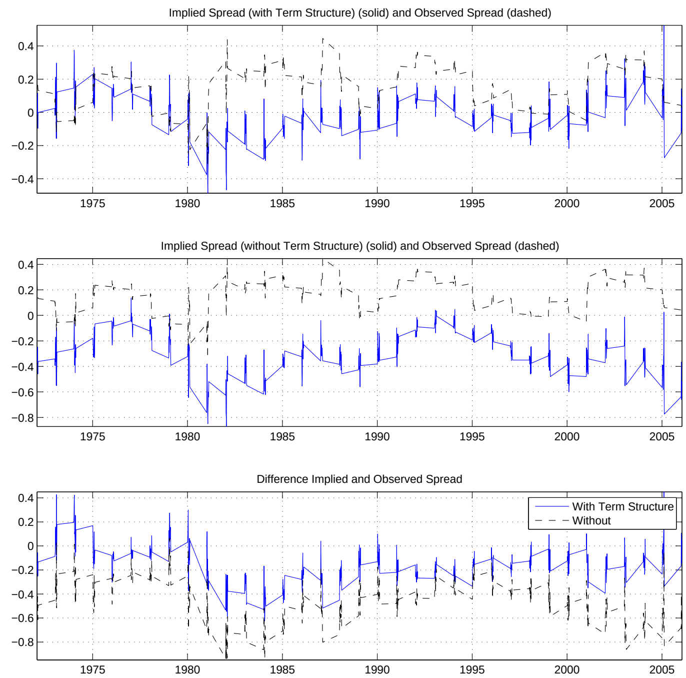
$$\sqrt{\sum_{t,\tau}(r_{\tau,t} - r_{\tau,t}^{obs})^2},$$

FIG. 8. Spread - Calibration 1



where $r_{\tau,t}$ is the model implied yield as given by (12) and $r_{\tau,t}^{obs}$ the observed yield. The first column specifies where we plug implied and where we plug actual volatilities into the model. If nothing else is mentioned, we use actual time-varying volatilities. "σ_{π^*} implied", for example, means that we use the implied time-varying volatility for the inflation target and actual time-varying volatilities for the other shocks. Not surprisingly, the fit of the model-implied yield curve is better if implied volatilities are used. It improves by a factor of 1.4 or higher (as given by the ratios 12.2/8.7, 14.4/10.0,...). Note that the models with constant implied volatilities even outperform models with time-varying actual volatilities. For the model with a cost-push shock, however, this difference is negligible. We further find that the first calibration provides a better fit than the second calibra-

FIG. 9. Spread - Calibration 2

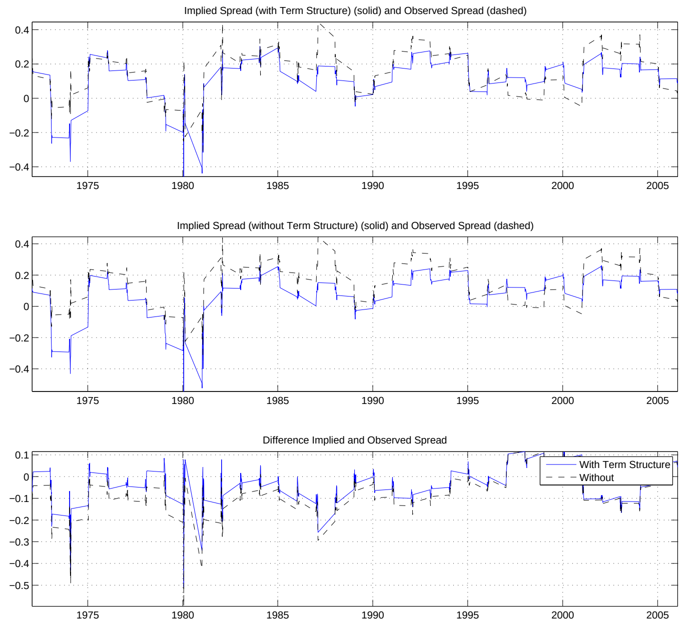


tion. Both calibrations, however, perform less well than the economy with a cost-push shock. The fit of the model with a cost-push shock is better by a factor of 2.2 or higher (leaving out the case constant actual volatilities). This confirms our previous conjecture. In a model without cost-push shock, the volatility of inflation target has to explain the volatility of observed inflation and can therefore not help to improve the fit of the implied term structure.

In addition, we have observed the following. It does not appear to matter whether we use implied or actual volatility for technology and/or government or the cost-push shock. However, we obtain a better fit if we use implied volatilities solely for the monetary policy related shocks.

Figures 8, 9 and 10 reflect the contents of Table 2. They compare the model implied yield slope with the observed term spread. The model implied slope turns out to be lower than the observed slope. This underestimation is obviously less pronounced if we use implied instead of the

FIG. 10. Spread - Cost-Push Shock



actual volatilities. More interestingly, in the cost-push economy the slope is quite well estimated independent of whether implied or actual volatilities are used.

6. CONCLUSION

We used yield curve data to estimate volatilities of structural shocks in a benchmark sticky price model. We assessed time-variation of the volatilities with a rolling data window of 60 months. We found that the sticky price model with a cost-push shock and time-varying actual volatilities, that is to say estimated solely with macro data, reproduced well the behavior of observed U.S. zero coupon bond yields. Without the cost-push shock, the fit was less satisfactory. This was reflected by the differences between implied and actual volatilities. In future research, more detailed models could be considered. An aim could be to relate the cost-push shock to structural explanations.

TABLE 2.

Square Root of Sum of Squared Differences Observed - Implied Yields

Volatility	Calibration 1	Calibration 2	Cost-push Shock
Constant actual volatilities	29.9	25.3	24.6
Constant implied volatilities	9.8	10.8	4.3
Varying actual volatilities	12.2	14.4	4.4
Varying implied volatilities	8.7	10.0	4.0
σ_{π^*} implied	9.6	10.5	4.3
σ_r implied	12.2	14.3	4.1
σ_a implied	12.2	14.5	4.4
σ_g implied	11.7	14.3	4.4
σ_{π^*}, σ_r implied	9.6	10.3	4.0
σ_a, σ_g implied	11.7	14.5	4.4
σ_u implied	-	-	4.4
σ_{π^*}, σ_r, σ_u implied	-	-	4.0

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