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Pricing Foreign Equity Options with Stochastic Correlation and Volatility

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A new class of foreign equity option pricing model is suggested that not only allows for the volatility but also for the correlation coefficient to vary stochastically over time. A modified Jacobi process is proposed to evaluate risk premium of the stochastic correlation, and a partial differential equation to price the correlation risk for the foreign equity has been set up, whose solution has been compared with the one with constant correlation. Since taking into account the stochastic volatility gives rise to more dimensions that produce more difficulty in numerical implementation of partial differential equation and Monte carlo, we figure out a series solution for pricing options under the correlation risk.

Key Words: Exotic option; Option pricing; Correlation risk; Portfolio; Random walk.
JEL Classification Numbers: G10, G12, G13, D81, E43.

1. INTRODUCTION

Option pricing model has traditionally employed the pioneer approach of Black and Scholes (1973) to determine the risk premium. Over the past few years, many people have been looking for pricing models which incorporate random volatility since empirical evidence, i.e., a variety of financial time series support the hypothesis of stochastic volatility. On the other hand, implied volatilities calculated using the Black-Scholes formula seem to change randomly over time. The fact we often see the smile of term structure of the implied volatility is due to the use of inappropriate measure for analysis, which can be accounted for by stochastic volatility approaches. Subsequently, many efforts have been devoted to solve the hard problem of finding the correct variable by quantitatively analyzing the impact of random motions of the volatility of assets (e.g., Hull and

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White, 1987, 1988; Heston, 1993; Ball and Roma, 1994). Generally, the problem is eliminated and the stochastic volatility model elucidates the deviations from constant implied volatility since the amount of persistence in the smile incorporates long-term memory in stochastic volatility. On the other hand, in the vast majority of financial and economic literature for multi-asset options, the correlation coefficient between any correlated variables has traditionally been assumed to be constant (e.g., see Balck and Scholes, 1973; Margrable, 1978; Garman, 1992). However, taking the long-term estimates of constant correlation may be misleading and would overestimate or underestimate the current correlation, consequently, it might cause serious problem in risk-taking, pricing, or hedging. In fact, a historical correlation should be used very carefully since generally the correlation might be more unstable than volatility. Moreover, another approach is to back out an implied correlation from the quoted price of the market instrument. The spirit behind this method is the same as with implied volatility, which might indicate us an estimation of the concept of stochastic correlation from market information. Therefore, to price multi-asset options, e.g., rainbow option, the foreign equity option, besides the stochastic volatilities, another random factor, i.e., stochastic correlation should be included as well. Recently, the market data analysis reveals the implied correlation deviates from the realized correlation, and figures out the non-zero correlation risk premium (see Buraschi, Porchia, and Trojani, 2006; Driessen, Maenhout, and Vilkov, 2006). Therefore, the evidence gives us confidence to investigate the model with structures of random correlation. On the other hand, although the large and rapidly growing literature deals with various types of exotic options (e.g., Taleb, 1997; Briys, Bellalah, Mai, and de Varenne, 1998; Kwok, 1998; Zhang, 1998; Baz and Chacko, 2004), the issue of the effect of stochastic correlation on the valuation of such options has not been expanded.

Currently, the equity-based derivative markets have become globalized. For example, Nikkei stock index options and futures are traded in Singapore, and, many foreign stocks are traded in the New York Stock Exchange. Trading of foreign equity derivatives always involves exchange rate uncertainty, but, sometimes, the trader could use the quanto options to avoid such a kind of risk. Quanto is mostly designed in currency-based markets with the price of one underlying foreign asset converted to domestic currency. We could not utilize the extended Black-Scholes formula to price quanto options since the exchange rate is generally correlated to the stock price. As pointed out in the above paragraph, the correlation is not fixed in actual world, and correlation risk exists not only in equity market but also in interest rate product market (see Buraschi, Cieslak and Trojani, 2007). Therefore, the correlation risk must be priced and the stochastic correlation should be included in the model for foreign equity options.

In Sec.2, the Jacobi process is utilized to describe the random motion of correlation coefficients and a partial differential equation for quanto options is derived. The numerical price for stochastic correlation is compared with the one for constant correlation. Sec.3 is devoted to the option with stochastic correlation coefficient and volatility. In Sec.4, taking a correlation option as the example, we analytically figured out a series solution of pricing model. In Sec.5, some discussions are made.

2. RISK-NEUTRAL PRICING OF OPTIONS WITH STOCHASTIC CORRELATION COEFFICIENT

Consider a more complex case for which the securities and their correlation coefficient are stochastic. To simplify the problem, in this section, we take the volatilities of security as constants. It is known that correlation may be connected to industrial production, to T-bill rates, to unanticipated inflation, namely, it seems that a varying correlation is just a business cycle indicator. But, in fact, after removing all business cycle effects carefully, the correlation risk still remains (see Driessen, Maenhout, and Vilkov, 2006). However, the correlation coefficient between two assets is not traded in the capital market, therefore, it is still necessary to figure out the portfolio to hedge the correlation risk by the Merton-Garman algorithm (see Merton, 1973).

To describe the stochastic correlation coefficient, we have two choices, i.e., one is the generalized autoregressive conditional heteroskedastic (GARCH)-type model (e.g. Engle, 1982; Scott, 1986; Golsten, Jagannathan, and Runkle, 1993) and the other is the continuous time approach. But since GARCH process has a nonlinear structure, the time aggregation properties of the GARCH models are not very convenient. Therefore, recently, another useful approach, i.e., Wishart autoregressive process (see Bru, 1991), is introduced to describe the changing correlation (see Gourieroux, Jasiak, and Sufana, 2004; Gourieroux and Sufana, 2004a, Gourieroux and Sufana, 2004b), which guarantees the variance-covariance matrix always is positive definite and might be useful for evaluating some options (see Fonseca, Grasselli and Tebaldi, 2005; Fonseca, Grasselli, and Tebaldi 2006). Furthermore, another suggested strategy would be to develop pricing models for products with the continuous-time version of stochastic correlation coefficient approach, and one could estimate their parameters with the discrete-time approximations and tests the specification on the discrete GARCH model.

To build the continuous time model for pricing the foreign equity options, we introduce two Geometric Brownian motions to describe the movement of the exchange rate S_d and the one of underlying foreign asset S_f in the real world as follows

$$\begin{aligned} dS_d &= \mu_d S_d dt + \sigma_1 S_d dW_1 \\ dS_f &= \mu_f S_f dt + \sigma_2 S_f dW_2, \end{aligned} \tag{1}$$

with a correlation coefficient between them

$$dW_1 dW_2 = \rho dt. \tag{2}$$

To simplify the problem, the volatilities of the two assets, σ_1 and σ_2 are both taken as constants, however, we assume the correlation coefficient of real world does a random walk, which could be written as

$$d\rho = (\bar{\rho} - \beta\rho)dt + \sigma_3\sqrt{(h-\rho)(\rho-f)}dW_3, \tag{3}$$

where, $\bar{\rho}-\beta\rho$ is a drift term, σ_3 is its volatility, $1 \geq h \geq f \geq -1$ and $h > \bar{\rho} > f$. Here, we need to select the parameters such as $(\bar{\rho}-\beta f) > \sigma_3^2(h-f)/2$ and $(\beta h-\bar{\rho}) > \sigma_3^2(h-f)/2$ to make sure ρ never cross over the bounds. It should be noted that the bound for correlation is $h \geq \rho \geq f$. Furthermore, we assume nonzero relationships between the price and correlation coefficient process, which read

$$\begin{aligned} dW_1 dW_3 &= \rho_1 dt \\ dW_2 dW_3 &= \rho_2 dt. \end{aligned} \tag{4}$$

h and f are the maximum and minimum value of ρ, which also should make the correlation matrix (elements: $\rho, \rho_1, \rho_2, 1$) for dW_1, dW_2, dW_3 positive definite.

Using Ito's Lemma, we obtain a three-dimensional stochastic differential equation for price of quanto option denoted by C

$$\begin{aligned} dC &= [\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 S_d^2 \frac{\partial^2 C}{\partial S_d^2} + \frac{1}{2}\sigma_2^2 S_f^2 \frac{\partial^2 C}{\partial S_f^2} + \rho\sigma_1\sigma_2 S_d S_f \frac{\partial^2 C}{\partial S_d \partial S_f} \\ &+ \sigma_1\sigma_3\rho_1 S_d\sqrt{(h-\rho)(\rho-f)}\frac{\partial^2 C}{\partial\rho\partial S_d} \\ &+ \sigma_2\sigma_3\rho_2 S_f\sqrt{(h-\rho)(\rho-f)}\frac{\partial^2 C}{\partial\rho\partial S_f} + \frac{1}{2}\sigma_3^2(h-\rho)(\rho-f)\frac{\partial^2 C}{\partial\rho^2}]dt \\ &+ \frac{\partial C}{\partial S_d}dS_d + \frac{\partial C}{\partial S_f}dS_f + \frac{\partial C}{\partial\rho}d\rho. \end{aligned} \tag{5}$$

To obtain the price of the option, following the Black-Scholes analysis, we consider two different options, $C_1(S_d, S_f, K_1, T_1)$ and $C_2(S_d, S_f, K_2, T_2)$ on the same underlying assets with different strike prices or maturities given by K_1, K_2, T_1 and T_2 respectively. In quanto option, we may use the portfolio

$$\Pi = C - \Gamma_d S_d - \Gamma_f S_d S_f, \tag{6}$$

to hedge the risks from the exchange rate and the foreign underlying asset, and another portfolio $\Pi_1 + \Gamma_1 \Pi_2$ to hedge the correlation risk. Using the Eq.(5), we get a series of equation to hedge the risks, which yields

$$\begin{aligned} \Gamma_1 &= -\frac{\partial C_1/\partial \rho}{\partial C_2/\partial \rho} \\ \Gamma_d &= \frac{\partial C}{\partial S_d} - \frac{S_f}{S_d}\frac{\partial C}{\partial S_f} \\ \Gamma_f &= \frac{1}{S_d}\frac{\partial C}{\partial S_f}. \end{aligned} \tag{7}$$

Using the above portfolio, the correlation risk could be eliminated, and the drift term of correlation process becomes $\bar{\rho} - \beta\rho + \bar{\lambda}(\rho, t)$ where $\bar{\lambda}(\rho, t)$ is a risk premium and could be defined as $\bar{\lambda}\rho$. Rewriting the drift term as $\bar{\rho} - g\rho$, we take the risk-neutral process of correlation as the Jacobi process

$$d\rho = (\bar{\rho} - g\rho)dt + \sigma_3\sqrt{(h-\rho)(\rho - f)}dW_3 \tag{8}$$

where $\bar{\rho}$ is related to the equilibrium value of correlation coefficient, i.e., $\frac{\bar{\rho}}{g}$, g is a positive parameter. Meanwhile, the risk-neutral asset prices become $\frac{dS_d}{S_d} = (r_d - r_f)dt + \sigma_1 dW_1$, $\frac{dS_f}{S_f} = r_f dt + \sigma_2 dW_2$, where r_d, r_f are domestic and foreign interest rate respectively. If measuring the foreign asset in domestic currency, the expected return in risk-neutral world is $r_d - (r_d - r_f) - \rho\sigma_1\sigma_2 = r_f - \rho\sigma_1\sigma_2$. Now, the constrained condition to make $h \geq \rho \geq f$ is changed as $(\bar{\rho} - gf) > \sigma_3^2(h-f)/2$ and $(gh - \bar{\rho}) > \sigma_3^2(h-f)/2$, which could be realized easily since $\bar{\lambda}$ is not very large. Consequently, ρ in the risk-neutral process can not cross over the either border. It should be emphasized the correlation matrix $\begin{pmatrix} 1 & \rho & \rho_1 \\ \rho & 1 & \rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}$ must be positive definite, whose determinant is zero or positive, i.e., $(1 - \rho_2^2 - \rho_1^2 + 2\rho\rho_1\rho_2 - \rho^2) \geq 0$. Then its solution is as follows

$$\rho_1\rho_2 - \sqrt{(1-\rho_1^2)(1-\rho_2^2)} \leq f \leq \rho \leq h \leq \rho_1\rho_2 + \sqrt{(1-\rho_1^2)(1-\rho_2^2)}. \tag{9}$$

Fixing all parameters, the risk-neutral pricing of a quanto option becomes

$$
\begin{aligned}
&\frac{\partial C}{\partial t} + (r_d - r_f)S_d\frac{\partial C}{\partial S_d} + (r_f - \rho\sigma_1\sigma_2)S_f\frac{\partial C}{\partial S_f} + (\bar{\rho} - g\rho)\frac{\partial C}{\partial \rho} \\
+\ & \frac{1}{2}\sigma_1^2 S_d^2\frac{\partial^2 C}{\partial S_d^2} + \frac{1}{2}\sigma_2^2 S_f^2\frac{\partial^2 C}{\partial S_f^2} + \rho\sigma_1\sigma_2 S_d S_f\frac{\partial^2 C}{\partial S_d \partial S_f} \\
+\ & \sigma_1\sigma_3\rho_1 S_d\sqrt{(h-\rho)(\rho-f)}\frac{\partial^2 C}{\partial\rho\partial S_d} + \sigma_2\sigma_3\rho_2 S_f\sqrt{(h-\rho)(\rho-f)}\frac{\partial^2 C}{\partial\rho\partial S_f} \\
+\ & \frac{1}{2}\sigma_3^2(h-\rho)(\rho-f)\frac{\partial^2 C}{\partial\rho^2} - r_d C = 0. \qquad (10)
\end{aligned}
$$

This equation is valid for any foreign equity option with underlying measured in foreign currency but paid in domestic one. Since we only need S_d to hedge, a solution independent of the exchange rate could be figured out. Rewriting the solution $C(S_d, S_f, t) = V(S_f, t)$, we get

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + (r_f - \rho\sigma_1\sigma_2)S_f\frac{\partial V}{\partial S_f} + (\bar{\rho} - g\rho)\frac{\partial V}{\partial \rho} + \frac{1}{2}\sigma_2^2 S_f^2\frac{\partial^2 V}{\partial S_f^2} \\
+\ & \sigma_2\sigma_3\rho_2 S_f\sqrt{(h-\rho)(\rho-f)}\frac{\partial^2 V}{\partial\rho\partial S_f} \\
+\ & \frac{1}{2}\sigma_3^2(h-\rho)(\rho-f)\frac{\partial^2 V}{\partial\rho^2} - r_d V = 0. \qquad (11)
\end{aligned}
$$

Then, using the payoff at expiration time $C(S_f, T) = \bar{S}_d \max[S_f(T) - K_f, 0]$ where $\bar{S}_d$ and K_f are a fixed exchange rate and the strike price respectively, the stochastic dynamics could give the price of quanto option. Certainly, stochastic equations could be solved by the Monte-Carlo method, whose accuracy and convergence unfortunately might be poor. Thus, we lay the Monte Carlo method aside temporarily and just use it as a supplementary tool in Sec. 4 only. Since we reduced a three-dimensional stochastic differential equation, i.e., Eqs.(1-3) to a two dimensional partial differential equation, the price of quanto option could be quickly solved as well as its Greeks are stably accessible. Subsequently, we solve Eq.(11) by finite difference method in this section, whose solution is analyzed by a series pricing formula derived in the next section. The Crank-Nicolson finite difference method is extremely popular for numerical solution of partial differential equation, whose main merits are its second order accuracy and stability. Discretizing Eq.(11), one could select the central difference in the Crank-Nicol method to increase the accuracy and stability of the solution. Then combining the terminal payoff function, i.e., $\bar{S}_d \max[S_f(T) - K_f, 0]$ we solve the price of the quanto option to avoid the correlation risk. The closed form solution for quanto option with a constant correlation yields

$$C = \bar{S}_d S_f e^{(r_f - r_d - \rho\sigma_1\sigma_2)\tau} N(d_1) - \bar{S}_d K_f e^{-r_d \tau} N(d_2), \tag{12}$$

where N is cumulative function, $\tau = T - t$, and

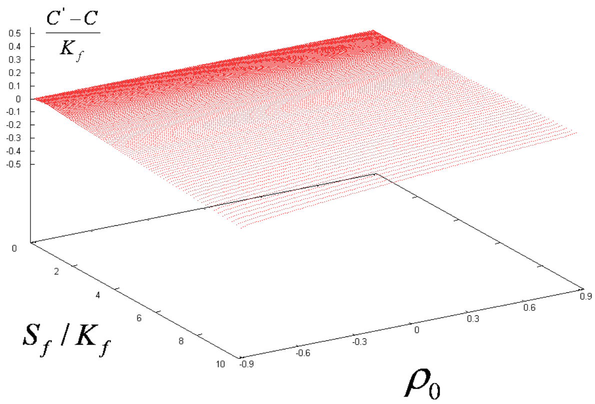
$$d_1 = \frac{\ln \frac{S_f}{K_f} + (r_f - \rho\sigma_1\sigma_2 + \frac{\sigma_2^2}{2})\tau}{\sigma_2\sqrt{\tau}}, d_2 = d_1 - \sigma_2\sqrt{\tau}. \tag{13}$$

The prices $C^{'}$ for stochastic correlation model and C for constant correlation could be computed respectively, and the difference $(C^{'}(\rho_0)-C(\rho_0))/K_f$ is plotted in Fig.1 where $\bar{S}_d = 1$, $r_f = 0.1$ $\sigma_1 = 1.0$, $\sigma_2 = 1.0$, $T - t = 0.15$, $\bar{\rho} = -0.01$, $\rho_1 = -\rho_2 = 0.01$, $g = 2.0$, $\sigma_3 = 1.0$ and $h = 0.9, f = -0.9$, ρ_0 is the initial correlation at time $t = 0$. It could be found the difference is small since the diffusion at $T = 0.15$ results in a weak deviation of ρ from the initial value ρ_0. Taking parameters same as in Fig.1 except $T = 0.30$, we compare the results with the price for constant coefficient in Fig.2, and could find larger price difference than in Fig. 1. But we should note that $(C^{'}(\rho_0) - C(\rho_0))/K_f$ for this option is large only around $\rho \sim \pm 0.9$ in Figs. 1 and 2. We note that the larger $S_f(t = 0)$, the bigger difference between $C^{'}(\rho_0) - C(\rho_0)$, which is exactly why we choose fixed exchange rate foreign equity call as example. In fact, other kinds of foreign equity option such as foreign equity call struck in foreign currency, and foreign call in domestic currency do not display such a phenomenon. In other words, their price difference due to correlation risk is almost zero and independent of ρ_0 as $S_f(t) >> K_f$ or $S_f(t) << K_f$. We will elucidate why in the next section.

3. THE STANDARD JACOBI DIFFERENTIAL EQUATION

It is known that the stochastic volatility is important as well as the correlation risk and they might have same source. For example, a firm's value can be decomposed as the net present value of all its forthcoming income with its asset minus its debt. Their components have different volatilities which cause the leverage related skew of the implied volatility. On the other hand, economic effects, e.g., anticipated central bank action, give rise to an interest rate skew of volatility, which is elucidated by the stochastic volatility. Then the same question arises for correlation coefficients. It is obvious that the correlation coefficients between those pairs of component in a firm's value must differ each other, which attributes to the stochastic correlation. The only difference between two cases is: to calibrate the market data of the skew or smile, we need to set up a comparatively strong correlation between the stochastic process of volatility and the one of equity. But, the correlation process could be independent or weakly connected to the

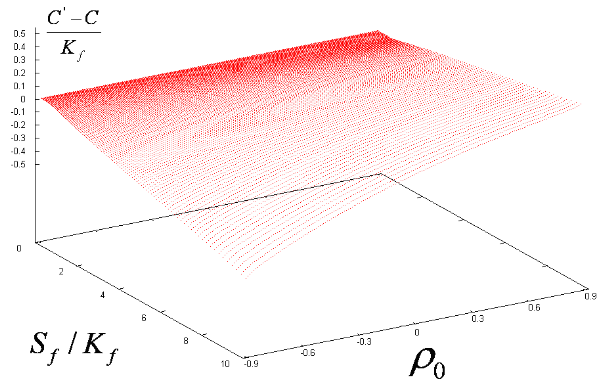
FIG. 1. The price difference $\frac{C'(\rho_0)-C(\rho_0)}{K_f}$ for $T = 0.15$. $C'(\rho_0)$ stands for the price of quanto option with the stochastic correlation coefficient, and $C(\rho_0)$ is the one with constant correlation coefficient, and K_f is the strike price. The parameters for C' are taken as $\bar{S}_d = 1$, $r_f = 0.1$ $\sigma_1 = 1.0$, $\sigma_2 = 1.0$, $T = 0.15$, $\bar{\rho} = -0.01$, $\rho_1 = -\rho_2 = 0.01$, $g = 2.0$, $\sigma_3 = 1.0$ and $h = 0.9, f = -0.9$. All parameters are same except the constant correlation is taken as ρ_0 for C where ρ_0 is the correlation of the initial time.



one of equity. Since generally, the empirical evidence shows that random correlation could move from 1 to -0.5 or even to -1 which suppresses ρ_1 and ρ_2 to make the correlation matrix positive definite, i.e., $\rho_1 \approx 0$ and $\rho_2 \approx 0$ (see Eq.(9)), we could use the pricing formula derived in this section for $\rho_1 = \rho_2 = 0$ with confidence.

Notice that we have not even introduced so far what stochastic volatility could be incorporated in this model. Certainly, we could generalize the model in Sec.2 to include the effect of stochastic volatilities of securities as well. Heston has considered the following process to describe the random walk of volatility (see Heston, 1993) $dP_k = (e_k - a_k P_k)dt + \gamma_k P_k^{\alpha_k} dV_k$ where $P_k = \sigma_k^2$, e_k and a_k are two positive parameters, γ_k represents the amplitude of change of volatility, α_k stands for a positive parameter less than 1, and dV_k is a standard Weiner process. Furthermore, we could use $d\rho_j = (\bar{\rho}_j - g_j \rho_j)dt - \sqrt{(h_j - \rho_j)(\rho_j - f_j)}dW_j$ to describe correlations in a set of underlying asset S_k where $1 \leq k \leq N$, and $1 \leq j \leq N(N-1)/2$. Then each pair of processes contributes a correlation coefficient as an element of the correlation matrix. It is very difficult to figure out a formula like

FIG. 2. The price difference $\frac{C'(\rho_0)-C(\rho_0)}{K_f}$ for $T = 0.30$. All parameters are same as in the figure 1 except $T = 0.30$.



Eq.(9) to guarantee this correlation matrix is positive definite. But a non-zero measure must exist for such a set of process since at least we could set each process as independent one even if highlighting the stochastic volatilities. Then, we could use the Heston process and the modified or standard Jacobi process to describe the random walks of volatility and correlation in N-dimensional assets, which could make the pricing model more sophisticated.

Generally, we should look for a closed form solution because of its convenience in practice. Regardless of that the closed form solution for this kind of stochastic process is currently unavailable, when choosing some special parameters, an approximate formula still could be derived. In last section, we recommend the Jacobi processes to describe the risk-neutral motion of correlation, i.e., $d\rho = (\bar{\rho} - g\rho)dt + \sigma_3\sqrt{(h-\rho)(\rho-f)}dW_3$. To simplify the problem, we select $\rho_1 = \rho_2 = 0$ to keep the correlation matrix definite positive. Now, the motion of correlation and the one of asset price are independent, then, using the Kolmogorov forward equation, we could solve the probability kernel for the correlation process easily. Therefore, the problem is how to determine the eigenvalue λ_n and eigenfunction $\psi_n(\rho)$

of the Markov generator of the process, which is

$$H = \frac{\sigma_3^2}{2}(h-\rho)(\rho-g)\frac{d^2}{d\rho^2} + (\bar{\rho} - g\rho)\frac{d}{d\rho}. \tag{14}$$

The eigenfunctions of operator H solve the equation

$$H\psi_n(\rho) = \lambda_n \psi_n(\rho), \tag{15}$$

which could be solved by variable separation method as follows

$$\frac{\sigma_3^2}{2}(h-\rho)(\rho-f)\frac{d^2\psi_n(\rho)}{d\rho^2} + (\bar{\rho} - g\rho)\frac{d\psi_n(\rho)}{d\rho} + \lambda_n\psi_n(\rho) = 0. \tag{16}$$

Introducing a transformation like $\rho = \frac{h-f}{2}\rho^{'} + \frac{h+f}{2}$, we rewrite the Eq.(16) as

$$\frac{\sigma_3^2}{2}(1-\rho^{'2})\frac{d^2\psi_n(\rho^{'})}{d\rho^{'2}} + (\frac{2\bar{\rho}}{h-f} - g\frac{h+f}{h-f} - g\rho^{'})\frac{d\psi_n(\rho^{'})}{d\rho^{'}} + \lambda_n\psi_n(\rho^{'}) = 0. \tag{17}$$

Then multiplying $\frac{2}{\sigma_3^2}$ in both sides, we get a standard Jacobi differential equation (see Szego, 1975)

$$(1-\rho^{'2})\frac{d^2\psi_n(\rho^{'})}{d\rho^{'2}} + \left[\frac{2}{\sigma_3^2}(\frac{2\bar{\rho} - hg - fg}{h-f}) - \frac{2}{\sigma_3^2}g\rho^{'}\right]\frac{d\psi_n(\rho^{'})}{d\rho^{'}} + \frac{2}{\sigma_3^2}\lambda_n\psi_n(\rho^{'}) = 0, \tag{18}$$

whose solutions yield

$$\lambda_n = -\frac{\sigma_3^2}{2}n(n + \frac{2g}{\sigma_3^2} - 1) \tag{19}$$

and

$$\psi_n(\rho^{'}) = \left[\frac{(2n + \frac{2g}{\sigma_3^2} - 1)\Gamma(n + \frac{2g}{\sigma_3^2} - 1)n!}{2^{\frac{2g}{\sigma_3^2}-1}\Gamma(n + \frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)})\Gamma(n + \frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)})}\right]^{1/2} P_n^{(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1, \frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1)}(\rho^{'}). \tag{20}$$

The Jacobi polynomials are as the following

$$\begin{aligned} & P_n^{(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1, \frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1)}(\rho^{'}) \\ = & \frac{(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)})_n}{n!} {}_2F_1\left(-n, n + \frac{2g}{\sigma_3^2} - 1; \frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}; \frac{1 - \frac{2\rho}{h-f} + \frac{h+f}{h-f}}{2}\right). \end{aligned} \tag{21}$$

where $(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)})_n = (\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)})(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}+1)(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}+2)\cdots(\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}+n)$ and ${}_2F_1$ is the hypergeometric polynomials. Basing on $(gh-\bar{\rho}) > \sigma_3^2(h-f)/2$ and $0 < h-f \leq 2$, we know $\frac{(2gh-2\bar{\rho})}{\sigma_3^2(h-f)} > 0$. Then the probability kernel for the Jacobi process, $P(\rho_0, \rho; t, T)$ could be written as

$$P(\rho_0, \rho; 0, \tau) = \sum_{n=0}^{\infty} e^{\lambda_n \tau}(1+\frac{h+f}{h-f}-\frac{2\rho}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1}(1-\frac{h+f}{h-f}+\frac{2\rho}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_n(\rho_0)\psi_n(\rho), \tag{22}$$

where $\tau = T - t$.

The probability kernel of whole system might be impossibly available, which raises difficulty to get the closed form solution. However, we still could derive a solution of a Taylor series expansion. First, we know the price of the option with two assets S_1 and S_2 at maturity T is determined by the terminal distribution of its payoff, which is denoted as $PO[S_1(T), S_2(T)]$. Then the option price at t yields

$$C(S_1(t), S_2(t), \rho_t, t) = e^{-r(T-t)}\int_0^{\infty}\int_0^{\infty} PO[S_1(T), S_2(T)]\omega[S_1(T), S_2(T)|S_1(t), S_2(t), \rho_t]dS_1(T)dS_2(T), \tag{23}$$

where $\omega[S_1(T), S_2(T)|S_1(t), S_2(t), \rho_t]$ is the conditional probability density function of $S_1(T), S_2(T)$ in the risk-neutral world given $S_1(t)$, $S_2(t)$ and ρ_t. Second, since the option price is determined by the terminal distribution of price process of the underlying asset instantaneously uncorrelated with the Jacobi process, we could define another independent variable, i.e., the averaged correlation coefficient during the life of the option,

$$\hat{\rho} = \frac{1}{\tau}\int_0^{\tau} \rho_t dt, \tag{24}$$

to solve the conditional probability density function in Eq.(23). Always, we could equally divide a $\hat{\rho}$ path into k pieces from 0 to τ, and each time piece is Δt. Consequently, it could be defined that S_1^i, S_2^i are asset price at the end of the ith period and their correlation is ρ_{i-1}. Then $[\ln(S_1^i/S_1^{i-1}), \ln(S_2^i/S_2^{i-1})]$ yield a multi-variate normal distribution. It is known that S_1, S_2 are instantaneously uncorrelated with the correlation process, which means the probability distribution of $[\ln(S_1^i/S_1^{i-1}), \ln(S_2^i/S_2^{i-1})]$ is conditioned on ρ_{i-1}. Basing on the Cholesky decomposition, we use $\sigma_1\Delta W_1^i$ and $\sigma_2(\rho_i\Delta W_1^i+\sqrt{1-\rho_i^2}\Delta W_2)$ to describe $[\ln(S_1^{i+1}/S_1^i), \ln(S_2^{i+1}/S_2^i)]$. It could be justified that covariance between $\ln(S_1^{i+1}/S_1^i)+\ln(S_1^i/S_1^{i-1})$ and $\ln(S_2^{i+1}/S_2^i)+\ln(S_2^i/S_2^{i-1})$ is $\sigma_1\sigma_2(\rho_i+\rho_{i-1})\Delta t$, the variance of asset one is $2\sigma_1^2\Delta t$, and the one of asset two is $\sigma_2^2(\rho_i^2+1-\rho_i^2+\rho_{i-1}^2+1-\rho_{i-1}^2)\Delta t = 2\sigma_2^2\Delta t$, which indicate the distribution of $[\ln(S_1^{i+1}/S_1^{i-1}), \ln(S_2^{i+1}/S_2^{i-1})]$ is normal with the correlation $\frac{\rho_i+\rho_{i-1}}{2}$. Hence, the probability distribution of $[\ln(S_1^k/S_1^0), \ln(S_2^k/S_2^0)]$ conditioned on the path follow by ρ is normal with a correlation $\hat{\rho}$, which depends on $\hat{\rho}$ only. On the other hand, if the stochastic dynamic generates an infinite number of paths that give the same averaged correlation $\hat{\rho}$, they must generate a same terminal distribution of asset price. Thus, using the formula for the conditional density function with several random variables, the conditional probability density function of terminal asset prices could be rewritten as

$$\begin{aligned} &\omega[S_1(T), S_2(T)|S_1(t), S_2(t), \rho_t)] \\ = &\int_f^h \Lambda[S_1(T), S_2(T)|S_1(t), S_2(t), \hat{\rho}]\epsilon[\hat{\rho}|S_1(t), S_2(t), \rho_t]d\hat{\rho}, \end{aligned} \tag{25}$$

where $\Lambda[S_1(T), S_2(T)|S_1(t), S_2(t), \hat{\rho}]$ means the class of path of S_1, S_2 conditional upon $\hat{\rho}$. Substituting the above equation into Eq.(23), the option prices turns to be

$$\begin{aligned} &C(S_1(t), S_2(t), \rho_t, t) \\ = \ &e^{-r(T-t)} \int_0^\infty \int_0^\infty \int_f^h PO[S_1(T), S_2(T)]\Lambda[S_1(T), S_2(T)|S_1(t), S_2(t), \hat{\rho}] \\ \times \ &\epsilon[\hat{\rho}|S_1(t), S_2(t), \rho_t]d\hat{\rho}dS_1(T)dS_2(T) \\ = \ &\int_f^h \{e^{-r(T-t)} \int_0^\infty \int_0^\infty PO[S_1(T), S_2(T)]\Lambda(S_1(T), S_2(T)|S_1(t), S_2(t), \hat{\rho}) \\ \times \ &dS_1(T)dS_2(T)\}\epsilon[\hat{\rho}|S_1(t), S_2(t), \rho_t]d\hat{\rho}. \end{aligned} \tag{26}$$

As mentioned before, the prices of the underlying asset follow geometric Brownian motions, which guarantees the class of path of S_1, S_2 conditional upon $\hat{\rho}$, i.e., $\Lambda[S_1(T), S_2(T)|S_1(t), S_2(t), \hat{\rho}]$, generates a standard lognormal distribution. Then in above equation, the inner integral produces nothing but the Black-Scholes formula with a varying correlation coefficient: the constant correlation coefficient is replaced by $\hat{\rho}$. Finally, the option value yields

$$C(S_1, S_2, \rho_t) = \int_f^h C_{BS}(\hat{\rho})\epsilon(\hat{\rho})d\hat{\rho}, \tag{27}$$

where C_{BS} stands for the Black-Scholes pricing formula with correlation $\hat{\rho}$. Although it is difficult to derive $\epsilon(\hat{\rho})$, remembering the kernel of the Jacobi process is available, nevertheless, we could obtain the moments of $\hat{\rho}$ and substitute them to expand the above formula. Consequently, the option pricing formula could be written in its Taylor series as follows

$$\begin{aligned}
& C(S_1(t), S_2(t), \rho_t) \\
= \; & C_{BS}(\overline{\hat{\rho}}) + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial \rho^2}|_{\overline{\hat{\rho}}}\mathrm{Var}(\hat{\rho}) + \frac{1}{6}\frac{\partial^3 C_{BS}}{\partial \rho^3}|_{\overline{\hat{\rho}}}\mathrm{Skew}(\hat{\rho}) + \cdots \\
= \; & C_{BS}(\overline{\hat{\rho}}) + \frac{1}{2}\frac{\partial^2 C_{BS}}{\partial \rho^2}|_{\overline{\hat{\rho}}}(\overline{\hat{\rho}^2} - (\overline{\hat{\rho}})^2) + \frac{1}{6}\frac{\partial^3 C_{BS}}{\partial \rho^3}|_{\overline{\hat{\rho}}}(\overline{\hat{\rho}^3} - 3\overline{\hat{\rho}^2}\cdot\overline{\hat{\rho}} + 2(\overline{\hat{\rho}})^3) + \cdots .
\end{aligned} \tag{28}$$

Now, the unsolved problem is to evaluate $\overline{\hat{\rho}}$, $\overline{\hat{\rho}^2}$, $\overline{\hat{\rho}^3}$, and so on. Using the evolution kernel of the risk-neutral Jacobi process $P(\rho_0, \rho; 0, t)$ to derive the expectation of $\hat{\rho}$, i.e., $E^Q(\frac{1}{\tau}\int_0^\tau \rho_t dt)$, it is easy to get $\overline{\hat{\rho}}$, which reads

$$\begin{aligned}
\overline{\hat{\rho}} \; = \; & \frac{1}{\tau}\int_f^h \int_0^\tau \rho P(\rho_0, \rho; 0, t) dt d\rho \\
= \; & \frac{1}{\tau}\int_f^h \int_0^\tau [\rho \sum_{n=0}^{\infty} e^{-\frac{\sigma_3^2}{2} n(n+\frac{2g}{\sigma_3^2}-1)\tau}(1 + \frac{h+f}{h-f} - \frac{2\rho}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1} \\
& \times(1 - \frac{h+f}{h-f} + \frac{2\rho}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_n(\rho_0)\psi_n(\rho)] dt d\rho \\
= \; & \sum_{n=0}^{\infty} \frac{1 - e^{-\frac{\sigma_3^2}{2} n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^2}{2} n(n+\frac{2g}{\sigma_3^2}-1)\tau}\psi_n(\rho_0)\int_f^h d\rho[\rho(1 + \frac{h+f}{h-f} - \frac{2\rho}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1} \\
& \times(1 - \frac{h+f}{h-f} + \frac{2\rho}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_n(\rho)].
\end{aligned} \tag{29}$$

It is a little tricky to derive $\overline{\hat{\rho}^2}$. First, we rewrite the expectation of $\hat{\rho}^2$ as $\frac{1}{\tau^2}E^Q[(\int_0^\tau \rho_{t_1} dt_1)(\int_0^\tau \rho_{t_2} dt_2)] = \frac{1}{\tau^2}E^Q[\int_0^\tau \int_0^\tau \rho_{t_1}\rho_{t_2} dt_1 dt_2]$, and divide the integrating domain into two parts, i.e., part $t_1 > t_2$ and the other $t_2 > t_1$, whose contributions to the final value are same. Second, selecting one part, and substituting

$$\begin{aligned}
& E^Q[\int_0^\tau \int_0^\tau \rho_{t_1}\rho_{t_2} dt_1 dt_2] \\
= \; & \frac{2}{\tau^2}\int_0^\tau dt_1 \int_f^h d\rho_x \rho_x P(\rho_0, \rho_x; 0, t_1)\int_0^{\tau - t_1} dt_2 \int_f^h d\rho_y \rho_y P(\rho_x, \rho_y; 0, t_2),
\end{aligned} \tag{30}$$

we get

$$\begin{aligned}
\overline{\hat{\rho}^2} &= \frac{2}{\tau^2}\int_f^h\{\int_0^\tau[\rho_x P(\rho_0,\rho_x;0,t_1)\int_f^h\rho_y\int_0^{\tau-t_1}P(\rho_x,\rho_y;0,t_2)dt_2d\rho_y]dt_1\}d\rho_x\\
&= \frac{2}{\tau^2}\sum_{n,m=0}^{\infty}[\frac{1-e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^4}{4}m(m+\frac{2g}{\sigma_3^2}-1)n(n+\frac{2g}{\sigma_3^2}-1)}\\
&- \frac{e^{-\frac{\sigma_3^2}{2}m(m+\frac{2g}{\sigma_3^2}-1)\tau}-e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^4}{4}m(m+\frac{2g}{\sigma_3^2}-1)(n(n+\frac{2g}{\sigma_3^2}-1)-m(m+\frac{2g}{\sigma_3^2}-1))}]\\
&\times \psi_n(\rho_0)\int_f^h\int_f^h d\rho_x d\rho_y[\rho_x(1+\frac{h+f}{h-f}-\frac{2\rho_x}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1}\\
&\times (1-\frac{h+f}{h-f}+\frac{2\rho_x}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_n(\rho_x)\psi_m(\rho_x)\rho_y(1+\frac{h+f}{h-f}-\frac{2\rho_y}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1}\\
&\times (1-\frac{h+f}{h-f}+\frac{2\rho_y}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_m(\rho_y)]. \qquad (31)
\end{aligned}$$

At last, following the same algorithms, we rewrite the expectation of $\hat{\rho}^3$ as $\frac{1}{\tau^3}E^Q[\int_0^\tau\int_0^\tau\int_0^\tau\rho_{t_1}\rho_{t_2}\rho_{t_3}dt_1dt_2dt_3]$ and divide the integrating domain into six parts, i.e., $t_3>t_2>t_1$, $t_3>t_1>t_2$, $t_2>t_1>t_3$, $t_2>t_3>t_1$, $t_1>t_3>t_2$, $t_1>t_2>t_3$, whose contributions to total value are same. Then, performing the following integration

$$\begin{aligned}
&E^Q[\int_0^\tau\int_0^\tau\int_0^\tau\rho_{t_1}\rho_{t_2}\rho_{t_3}dt_1dt_2dt_3]\\
&= \frac{6}{\tau^3}\int_f^h d\rho_x\int_0^\tau dt_1\rho_x P(\rho_0,\rho_x;0,t_1)\int_f^h d\rho_y\int_0^{\tau-t_1}dt_2\rho_y P(\rho_x,\rho_y;0,t_2)\\
&\times \int_f^h d\rho_z\int_0^{\tau-t_2-t_1}dt_3\rho_z P(\rho_y,\rho_z;0,t_3), \qquad (32)
\end{aligned}$$

and substituting it, we get

$$
\begin{aligned}
&\overline{\hat{\rho}^3} \\
&= \frac{6}{\tau^3} \sum_{\substack{n=0\\ m=0,l=0}}^{\infty} [\frac{e^{-\frac{\sigma_3^2}{2}m(m+\frac{2g}{\sigma_3^2}-1)\tau} - e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^6}{8}l(l+\frac{2g}{\sigma_3^2}-1)(m(m+\frac{2g}{\sigma_3^2}-1)-l(l+\frac{2g}{\sigma_3^2}-1))(n(n+\frac{2g}{\sigma_3^2}-1)-m(m+\frac{2g}{\sigma_3^2}-1))} \\
&- \frac{e^{-\frac{\sigma_3^2}{2}l(l+\frac{2g}{\sigma_3^2}-1)\tau} - e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^6}{8}l(l+\frac{2g}{\sigma_3^2}-1)(m(m+\frac{2g}{\sigma_3^2}-1)-l(l+\frac{2g}{\sigma_3^2}-1))(n(n+\frac{2g}{\sigma_3^2}-1)-l(l+\frac{2g}{\sigma_3^2}-1))} \\
&- \frac{e^{-\frac{\sigma_3^2}{2}m(m+\frac{2g}{\sigma_3^2}-1)\tau} - e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^6}{8}l(l+\frac{2g}{\sigma_3^2}-1)m(m+\frac{2g}{\sigma_3^2}-1)(n(n+\frac{2g}{\sigma_3^2}-1)-m(m+\frac{2g}{\sigma_3^2}-1))} \\
&+ \frac{1-e^{-\frac{\sigma_3^2}{2}n(n+\frac{2g}{\sigma_3^2}-1)\tau}}{\frac{\sigma_3^6}{8}l(l+\frac{2g}{\sigma_3^2}-1)m(m+\frac{2g}{\sigma_3^2}-1)n(n+\frac{2g}{\sigma_3^2}-1)}]\psi_n(\rho_0) \\
&\times \int_f^h \int_f^h \int_f^h d\rho_x d\rho_y d\rho_z [\rho_x(1+\frac{h+f}{h-f}-\frac{2\rho_x}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1} \\
&\times (1-\frac{h+f}{h-f}+\frac{2\rho_x}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_n(\rho_x)\psi_m(\rho_x) \\
&\times \rho_y(1+\frac{h+f}{h-f}-\frac{2\rho_y}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1}(1-\frac{h+f}{h-f}+\frac{2\rho_y}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_m(\rho_y)\psi_l(\rho_y) \\
&\times \rho_z(1+\frac{h+f}{h-f}-\frac{2\rho_z}{h-f})^{\frac{2gh-2\bar{\rho}}{\sigma_3^2(h-f)}-1}(1-\frac{h+f}{h-f}+\frac{2\rho_z}{h-f})^{\frac{2\bar{\rho}-2gf}{\sigma_3^2(h-f)}-1}\psi_l(\rho_z)]. \qquad (33)
\end{aligned}
$$

Basing on the fact that the kurtosis of $\hat{\rho}$ is generally much less than 0.001 because of $|\rho_t| \leq 1$, absolute value of the fourth term in Eq.(28) is basically much less than one basis point. Thus, it is not necessary to derive $\overline{\hat{\rho}^4}$ and even higher order terms. We testify Eqs.(29,31,33) and compare the computation result with the one from the Monte Carlo method in Tables 1 and 2. Remembering that this Monte Carlo simulation is just one-dimensional and should have high accuracy, it is not surprising to observe that the numerical outcomes from two methods in Tables 1 and 2 are almost same. We need to point out that in Eqs.(29,31,33), the first couple of eigenvalue and eigenfunctions generally could guarantee very good accuracy since the eigenvalue $\lambda_n \sim -n^2$ and the exponentially decaying factor $e^{\lambda_n \tau}$ suppresses most of terms in Eqs.(29,31,33). Moreover, the first inner integration in Eqs.(29,31,33) could be performed analytically, which makes these equations more efficient.

An important subject is to analyze the numerical option price within the framework of Eq.(28). Substituting Black-Scholes solution of quanto option into Eq.(28), immediately, we could understand what happens in Figs. (1) and (2). Since at $\rho_0 \sim -0.9$, $\overline{\hat{\rho}}$ must be larger than the initial

TABLE 1.

$\overline{\hat{\rho}}, \overline{\hat{\rho}^2}$ and $\overline{\hat{\rho}^3}$ for $h = -f = 0.8, \bar{\rho} = 0.5, g = 1.7, \sigma_3 = 1.0, T - t = 5.0, \rho_0 = 0.6$

Method	$\overline{\hat{\rho}}$	$\overline{\hat{\rho}^2}$	$\overline{\hat{\rho}^3}$
Eqs (29,31,33)	0.3298124	0.1320751	0.0571973
Monte Carlo	0.3296324	0.1318939	0.0572296

TABLE 2.

$\overline{\hat{\rho}}, \overline{\hat{\rho}^2}$ and $\overline{\hat{\rho}^3}$ for $h = -f = 0.8, \bar{\rho} = -0.5, g = 1.7, \sigma_3 = 1.0, T - t = 5.0, \rho_0 = 0.6$

Method	$\overline{\hat{\rho}}$	$\overline{\hat{\rho}^2}$	$\overline{\hat{\rho}^3}$
Eqs (29,31,33)	-0.1887623	0.0611409	-0.0198942
Monte Carlo	-0.1896188	0.0614482	-0.0199905

correlation coefficient due to the almost zero equilibrium value of correlation, and at $\rho_0 \sim 0.9$, $\overline{\hat{\rho}}$ must be smaller than the initial one, we could know that from Eq.(28), the factor $e^{(r_f - r_d - \overline{\hat{\rho}}\sigma_1\sigma_2)\tau}$ in the zeroth order term determines that the price with stochastic correlation is larger than the price with constant one at $\rho_0 \sim 0.9$ or vice versus at $\rho_0 \sim -0.9$. On the other hand, $S_f e^{(r_f - r_d - \overline{\hat{\rho}}\sigma_1\sigma_2)\tau} N(d_1)$ in Eq.(12) implies that even $K_f << S_f(t=0)$, the price difference never vanishes and should be proportional to $S_f(t=0)$ as we could find exactly in Figs.(1) and (2). In fact, in other kinds of foreign currency option, the correlation risk is not so important as the spot price of underlying assets is very larger than the strike price. Taking foreign equity call struck in domestic currency as an example, whose terminal pay off is $Max(S_d S_f - K_d, 0)$, its Black-Scholes solution with constant correlation ρ between underlying asset and the exchange rate reads

$$C(S_f, \tau) = S_d S_f N(d_1) - K_d e^{-r_d \tau} N(d_2), \tag{34}$$

where K_d is K_d units of domestic currency for one unit of strike price of the underlying asset, and

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_f}{K_d} + [-r_d + 0.5(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\tau]}{\sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\tau}}, \\ d_2 &= d_1 - \sqrt{(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)\tau}. \end{aligned} \tag{35}$$

Then, we could immediately find a ρ-independent price $C \approx S_d S_f$ for $S_f >> K_d$, which implies no ρ sensitivity in Eqs.(28,34) for the option deep in the money. Following the same approach, we could justify as well no ρ sensitivity for foreign equity call struck in domestic currency when $S_f << K_d$, which indicates that the correlation risk is important only in

$S_f \sim K_d$ region. Then one could understand why we choose quanto option as an example to illustrate the price difference which is proportional to the value of underlying asset.

Another interesting issue is the pricing formula for the number of underlying assets $N >> 1$. For example, the problem is how to derive the correlation risk premium of a option involved in many underlying assets, such as index or basket option involved in three or more underlying assets. We take $N = 3$ as an example to present the general solution. First, we define three geometric Brownian motions to describe the price motion of three underlying assets. In three geometric Brownian motions, the diffusion terms, i.e., dW_1, dW_2 and dW_3 have the correlation relationships $dW_1 dW_2 = \rho_{12} dt$, $dW_1 dW_3 = \rho_{13} dt$, and $dW_2 dW_3 = \rho_{23} dt$ respectively. Second, we assume those correlation coefficients vary over time and follow three independent Jacobi processes. Then, defining $\hat{\rho}_{12}$, $\hat{\rho}_{13}$ and $\hat{\rho}_{23}$ like in Eq.(24), and figuring out their moments, we could expand the option price in its Taylor series as follows

$$
\begin{aligned}
& C(S_1(t), S_2(t), S_3(t), \rho_{12}(t), \rho_{13}(t), \rho_{23}(t)) \qquad (36)\\
= & \sum_{j=0}^{\infty} \{ \frac{1}{j!} E^Q [(\hat{\rho}_{12} - \bar{\hat{\rho}}_{12}) \frac{\partial}{\partial \rho_{12}} + (\hat{\rho}_{13} - \bar{\hat{\rho}}_{13}) \frac{\partial}{\partial \rho_{13}} + (\hat{\rho}_{23} - \bar{\hat{\rho}}_{23}) \frac{\partial}{\partial \rho_{23}}]^j \\
\times & C_{BS}(S_1(t), S_2(t), S_3(t), \rho_{12}(t), \rho_{13}(t), \rho_{23}(t))|_{\rho_{12} = \bar{\hat{\rho}}_{12}, \rho_{13} = \bar{\hat{\rho}}_{13}, \rho_{23} = \bar{\hat{\rho}}_{23}} \}
\end{aligned}
$$

This formula could be easily generalized to the option with more than three underlying assets.

4. OPTIONS WITH STOCHASTIC VOLATILITY AND CORRELATION

As pointed out in the proceeding sections, the stochastic volatility has an important impact on the option price. If wanting to understand the market evolution completely, e.g., the volatility smile, we have to take into account the stochastic volatility. But the stochastic volatility sets up more difficulties for us to get an explicit pricing formula. To simplify the problem, we select a correlation option as an example.

In an European call on a single asset, what the trader could utilize is just one spread between the terminal price of asset and the strike. If wanting to utilize two spreads simultaneously, the trade might need to use a tool looking like a two-spread analog of an European call or put option. Besides the product option giving traders a right to buy or sell the product of two underlying assets for the strike, one could design another option to allow traders to capture an opportunity to trade two spreads in assets jointly, whose payoff is $Max[S_1(T)-K_1,0]\times Max[S_2(T)-K_2,0]$ (see Bakishi and Madan, 2000).

Selecting this kind of option as an example, we could investigate an important issue, i.e., pricing option with stochastic volatility and correlation. First, we suppose the trader could use the correlation option to bet the spreads in the foreign exchange rate or asset. Second, no foreign exchange rate or asset keeps constant volatility, and all stochastic volatilities share one random source. Consequently, the dynamics with stochastic volatility could be selected as follows (see Jegadeesh and Tuckman, 2000; Bakishi and Madan, 2000)

$$\begin{aligned}\frac{dS_1}{S_1} &= (r_d-r_f)dt+\sigma_1\sqrt{Q}dW_1,\\ \frac{dS_2}{S_2} &= (r_d-r_f)dt+\sigma_2\sqrt{Q}dW_2,\\ dQ &= (\theta_q-k_qQ)dt+\sigma_q\sqrt{Q}dW_q,\end{aligned}\tag{37}$$

where Q represents a volatility factor, r_d, r_f are domestic and foreign interest rate respectively, we specify $dW_1dW_2=\rho dt$, $dW_1dW_q=\bar{\rho}_1dt$, and $dW_2dW_q=\bar{\rho}_2dt$. This ρ-constant dynamics values the correlation option $C(S_1,S_2,Q_0,\tau)$ as

$$\begin{aligned}&C(S_1,S_2,Q_0,\tau)=\\ &\frac{e^{-\zeta_1\ln K_1-\zeta_2\ln K_2}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Re[\frac{e^{-i(\phi_1\ln K_1+\phi_2\ln K_2)-r_d\tau}G(\tau;\chi_1,\chi_2)}{(\zeta_1+i\phi_1)(\zeta_1+1+i\phi_1)(\zeta_2+i\phi_2)(\zeta_2+1+i\phi_2)}]d\phi_2d\phi_1,\end{aligned}\tag{38}$$

where Q_0 is initial volatility, ζ_1 and ζ_2 are chosen as two real numbers to define $\chi_1=\phi_1-(1+\zeta_1)i,\chi_2=\phi_2-(1+\zeta_2)i$, and the characteristic function is

$$\begin{aligned}G(\tau;\chi_1,\chi_2) &= \exp[i\chi_1\ln S_1(0)+i\chi_2\ln S_2(0)+i(\chi_1+\chi_2)(r_d-r_f)\tau\\ &+ \frac{2\eta(1-e^{-\theta\tau})}{2\theta-(\theta-\varphi)(1-e^{-\theta\tau})}Q_0\\ &- \frac{2\theta_q}{\sigma_q^2}\ln(1-\frac{(\theta-\varphi)(1-e^{-\theta\tau})}{2\theta})-\frac{\theta_q}{\sigma_q^2}(\theta-\varphi)\tau],\end{aligned}\tag{39}$$

with

$$\begin{aligned}\varphi &= k_q-i(\bar{\rho}_1\sigma_1\chi_1+\bar{\rho}_2\sigma_2\chi_2)\sigma_q\\ \theta &= \sqrt{\varphi^2-2\sigma_q^2\eta}\\ \eta &= -0.5(\sigma_1^2\chi_1^2+\sigma_2^2\chi_2^2+2\rho\sigma_1\sigma_2\chi_1\chi_2)-0.5i(\sigma_1^2\chi_1+\sigma_2^2\chi_2).\end{aligned}\tag{40}$$

Having defined an independent Jacobi process to describe the random motion of correlation ρ, we could use Eq.(28) to derive the price of the option with stochastic volatility and correlation. Furthermore, the sensitivity of option price versus the correlation is

$$\frac{\partial G}{\partial \rho} = (-2\frac{\theta_q X_2}{\sigma_q^2 X_1} + \frac{\theta_q \tau X_0}{\theta} + Q_0 \frac{X_4}{X_3^2})G \tag{41}$$

with

$$\begin{aligned}
X_0 &= -\sigma_1\sigma_2\chi_1\chi_2 \\
y_0 &= -\sigma_q^2 X_0(1-e^{-\theta\tau})/\theta - (\theta-\varphi)\tau\sigma_q^2 X_0 e^{-\theta\tau}/\theta \\
X_1 &= 1 - \frac{(\theta-\varphi)(1-e^{-\theta\tau})}{2\theta} \\
X_2 &= -0.5\frac{\sigma_q^2 X_0(\theta-\varphi)(1-e^{-\theta\tau})}{\theta^3} - 0.5\frac{y_0}{\theta} \\
X_3 &= 2\theta - (\theta-\varphi)(1-e^{-\theta\tau}) \\
X_4 &= 2X_0(1-e^{-\theta\tau})X_3 - 2\eta\tau\sigma_q^2 X_0 X_3 e^{-\theta\tau}/\theta \\
&+ 2\eta(1-e^{-\theta\tau})(2\sigma_q^2 X_0/\theta + y_0).
\end{aligned} \tag{42}$$

The second order is

$$\begin{aligned}
\frac{\partial^2 G}{\partial \rho^2} &= [-2\theta_q(X_5X_1 - X_2^2)/(\sigma_q^2 X_1^2) + \theta_q\sigma_q^2\tau X_0^2/\theta^3 + X_6 Q_0/X_3^2 \\
&+ 4\sigma_q^2 X_0 X_4 Q_0/(X_3^3\theta) + 2X_4 y_0 Q_0/X_3^3]G \\
&+ (-2\theta_q X_2/(\sigma_q^2 X_1) + \theta_q\tau X_0/\theta + X_4 Q_0/X_3^2)\frac{\partial G}{\partial \rho}
\end{aligned} \tag{43}$$

with

$$\begin{aligned}
X_5 &= -\sigma_q^2 X_0 y_0/\theta^3 - 0.5y_1/\theta - \frac{3}{2}\sigma_q^4 X_0^2(\theta-\varphi)(1-e^{-\theta\tau})/\theta^5 \\
X_6 &= -2X_0^2 X_3\tau\sigma_q^2 e^{-\theta\tau}/\theta - 2X_0(1-e^{-\theta\tau})(2\sigma_q^2 X_0/\theta + y_0) \\
&- 2\tau\sigma_q^2 X_0^2 X_3 e^{-\theta\tau}/\theta + 2\eta\tau\sigma_q^2 X_0 e^{-\theta\tau}(2\sigma_q^2 X_0/\theta + y_0)/\theta \\
&- 2\eta\tau^2\sigma_q^4 X_0^2 X_3 e^{-\theta\tau}/\theta^2 - 2\eta\tau\sigma_q^4 X_0^2 X_3 e^{-\theta\tau}/\theta^3 \\
&+ 2X_0(1-e^{-\theta\tau})(2\sigma_q^2 X_0/\theta + y_0) + 2\eta X_7 \\
X_7 &= -2\tau\sigma_q^4 X_0^2 e^{-\theta\tau}/\theta^2 - \tau\sigma_q^2 X_0 y_0 e^{-\theta_1\tau}/\theta + (1-e^{-\theta\tau})(2\sigma_q^4 X_0^2/\theta^3 + y_1) \\
y_1 &= \tau\sigma_q^4 X_0^2 e^{-\theta\tau}/\theta^2 - \sigma_q^4 X_0^2(1-e^{-\theta\tau})/\theta^3 + \tau\sigma_q^4 X_0^2 e^{-\theta\tau}/\theta^2 \\
&- (\theta-\varphi)\tau^2\sigma_q^4 X_0^2 e^{-\theta\tau}/\theta^2 - (\theta-\varphi)\tau\sigma_q^4 X_0^2 e^{-\theta\tau}/\theta^3.
\end{aligned} \tag{44}$$

The third order is

$$
\begin{aligned}
\frac{\partial^3 G}{\partial \rho^3} &= [-2\theta_q X_8/(\sigma_q^2 X_1) + 2\theta_q X_5 X_2/(\sigma_q^2 X_1^2) + 4\theta_q X_2 X_5/(\sigma_q^2 X_1^2) \\
&- 4\theta_q X_2^3/(\sigma_q^2 X_1^3) + 3\theta_q \tau \sigma_q^4 X_0^3/\theta^5 \\
&+ X_9 Q_0/X_3^2 + 2X_6(2\sigma_q^2 X_0/\theta + y_0)Q_0/X_3^3 + 4\sigma_q^2 X_0 X_6 Q_0/(X_3^2\theta) \\
&+ 12\sigma_q^2 X_0 X_4(2\sigma_q^2 X_0/\theta + y_0)Q_0/(X_3^4\theta) + 4\sigma_q^4 X_0^2 X_4 Q_0/(X_3^3\theta^3) \\
&+ 2X_6 y_0 Q_0/X_3^3 + 2X_4 y_1 Q_0/X_3^3 + 6X_4 y_0(2\sigma_q^2 X_0/\theta + y_0)Q_0/X_3^4]G \\
&+ 2[-2\theta_q X_5/(\sigma_q^2 X_1) + 2\theta_q X_2^2/(\sigma_q^2 X_1^2) + \theta_q \tau \sigma_q^2 X_0^2/\theta^3 \\
&+ X_6 Q_0/X_3^2 + 4\sigma_q^2 X_0 X_4 Q_0/(X_3^3\theta) + 2X_4 y_0 Q_0/X_3^3]\frac{\partial G}{\partial \rho} \\
&+ [-2\theta_q X_2/(\sigma_q^2 X_1) + \theta_q \tau X_0/\theta + X_4 Q_0/X_3^2]\frac{\partial^2 G}{\partial \rho^2} \qquad (45)
\end{aligned}
$$

with

$$
\begin{aligned}
X_8 &= -y_1\sigma_q^2 X_0/\theta^3 - \frac{9}{2}y_0\sigma_q^4 X_0^2/\theta^5 - \frac{1}{2}y_2/\theta - \frac{1}{2}\sigma_q^2 X_0 y_1/\theta^3 \\
&- \frac{15}{2}\sigma_q^6 X_0^3(\theta - \varphi)(1 - e^{-\theta\tau})/\theta^7 \\
X_9 &= 6\tau\sigma_q^2 X_0^2(2\sigma_q^2 X_0/\theta + y_0)e^{-\theta\tau}/\theta - 6\tau^2\sigma_q^4 X_0^3 X_3 e^{-\theta\tau}/\theta^2 \\
&- 6\tau\sigma_q^4 X_0^3 X_3 e^{-\theta\tau}/\theta^3 + 4\eta\tau^2\sigma_q^4 X_0^2 e^{-\theta\tau}(2\sigma_q^2 X_0/\theta + y_0)/\theta^2 \\
&+ 2\eta\tau\sigma_q^2 X_0 e^{-\theta\tau}(2\sigma_q^4 X_0^2/\theta^3 + y_1)/\theta + 4\eta\tau\sigma_q^4 X_0^2 e^{-\theta\tau}(2\sigma_q^2 X_0/\theta + y_0)/\theta^3 \\
&- 2\eta\tau^3\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^3 - 6\eta\tau^2\sigma_q^6 X_0^3 e^{-\theta\tau} X_3/\theta^4 \\
&- 6\eta\tau\sigma_q^6 X_0^3 e^{-\theta\tau} X_3/\theta^5 + 2x_0 X_7 + 2\eta X_{10} \\
X_{10} &= -2\tau^2\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^3 - 4\tau\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^4 - \tau\sigma_q^2 X_0 y_1 e^{-\theta\tau}/\theta \\
&- \tau^2\sigma_q^4 X_0^2 y_0 e^{-\theta\tau}/\theta^2 - \tau\sigma_q^4 X_0^2 y_0 e^{-\theta\tau}/\theta^3 \\
&- \tau\sigma_q^2 X_0(2\sigma_q^4 X_0^2/\theta^3 + y_1)e^{-\theta\tau}/\theta + (1 - e^{-\theta\tau})(6\sigma_q^6 X_0^3/\theta^5 + y_2) \\
y_2 &= 3\tau^2\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^3 + 6\tau\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^4 \\
&- 3\sigma_q^6 X_0^3(1 - e^{-\theta\tau})/\theta^5 - (\theta - \varphi)\tau^3\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^3 \\
&- 3(\theta - \varphi)\tau^2\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^4 - 3(\theta - \varphi)\tau\sigma_q^6 X_0^3 e^{-\theta\tau}/\theta^5. \qquad (46)
\end{aligned}
$$

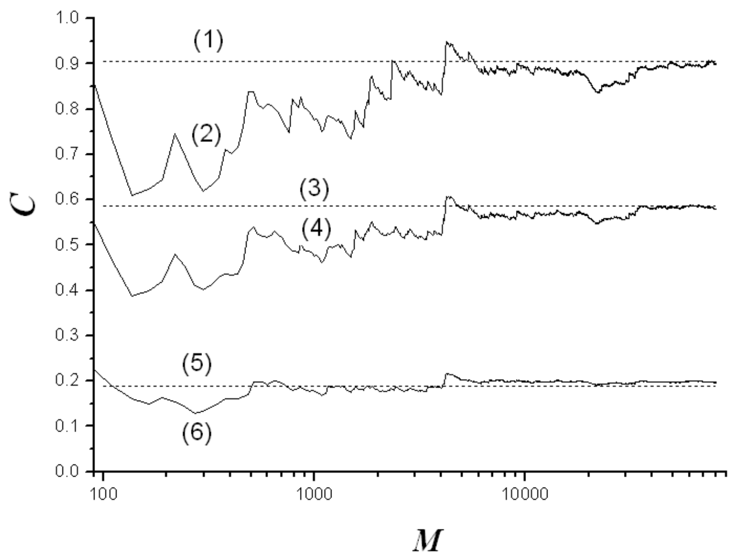
Substituting $\frac{\partial G}{\partial \rho}$, $\frac{\partial^2 G}{\partial \rho^2}$, and $\frac{\partial^3 G}{\partial \rho^3}$, and utilizing the algorithm introduced in Sec. 3 and Eqs.(29,31,33) for $\overline{\hat{\rho}}, \overline{\hat{\rho}^2}, \overline{\hat{\rho}^3}$, we could arrive at the final formula for pricing the option with stochastic volatility and correlation

$$\begin{aligned}
& C(S_1, S_2, Q_0, \rho_0) \\
= \ & \frac{e^{-\zeta_1 \ln K_1 - \zeta_2 \ln K_2}}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Re[\frac{e^{-i(\phi_1 \ln K_1 + \phi_2 \ln K_2 - ir_d\tau)} G(\tau; \chi_1, \chi_2)}{(\zeta_1 + i\phi_1)(\zeta_1 + 1 + i\phi_1)(\zeta_2 + i\phi_2)(\zeta_2 + 1 + i\phi_2)}] d\phi_2 d\phi_1 \\
+ \ & \frac{1}{2}\frac{e^{-\zeta_1 \ln K_1 - \zeta_2 \ln K_2}}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Re[\frac{e^{-i(\phi_1 \ln K_1 + \phi_2 \ln K_2 - ir_d\tau)} \frac{\partial^2 G(\tau;\chi_1,\chi_2)}{\partial \rho^2}|_{\overline{\hat{\rho}}}}{(\zeta_1 + i\phi_1)(\zeta_1 + 1 + i\phi_1)(\zeta_2 + i\phi_2)(\zeta_2 + 1 + i\phi_2)}] d\phi_2 d\phi_1 \\
\times \ & (\overline{\hat{\rho}^2} - (\overline{\hat{\rho}})^2) \\
+ \ & \frac{1}{6}\frac{e^{-\zeta_1 \ln K_1 - \zeta_2 \ln K_2}}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Re[\frac{e^{-i(\phi_1 \ln K_1 + \phi_2 \ln K_2 - ir_d\tau)} \frac{\partial^3 G(\tau;\chi_1,\chi_2)}{\partial \rho^3}|_{\overline{\hat{\rho}}}}{(\zeta_1 + i\phi_1)(\zeta_1 + 1 + i\phi_1)(\zeta_2 + i\phi_2)(\zeta_2 + 1 + i\phi_2)}] d\phi_2 d\phi_1 \\
\times \ & (\overline{\hat{\rho}^3} - 3\overline{\hat{\rho}^2} \cdot \overline{\hat{\rho}} + 2(\overline{\hat{\rho}})^3) + \cdots . \qquad (47)
\end{aligned}$$

Although the above formula is just for two-asset option, in fact, substituting a stochastic volatility pricing formula of index or basket option into Eq.(36), we could get the volatility and correlation risk pricing formula involving three or more underlying assets.

Before testifying the numerical method, in Eq.(37) model, we always can normalize the terminal payoff of the correlation as $K_1K_2Max[\frac{S_1(T)}{K_1} - 1] \times Max[\frac{S_2(T)}{K_2} - 1]$. Then taking $K_1 = K_2 = 1$ is convenient, which does not lose any generality at all. We use the Monte Carlo method and Eq.(38) to compute the price of Eq.(37) model with constant ρ. In Sec. 3, we find the numerical outcome from one-dimensional Monte Carlo matches Eqs.(29,31,33) well. But to compute Eq.(37) with constant ρ, the Monte Carlo must be multi-dimensional, which unfortunately lessens its accuracy. In lines (1) and (2) of Fig.3, taking $K_1 = K_2 = 1$, $S_1(0) = 1.1$, $S_2(0) = 1.2$ $\tau = 5$, $r_d = 0.1$, $r_f = 0.05$, $\rho = 0.6$, $\bar{\rho}_1 = \bar{\rho}_2 = -0.2$, $Q_0 = 0.8$ $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\theta_q = 0.5$, $\kappa_q = 1.0$, and $\sigma_q = 0.5$, the semi-closed form solution, Eq.(38) for constant $\rho = 0.6$ gives the price 0.905, but the asymptotic price for constant $\rho = 0.6$ from the Monte Carlo method is 0.899. Taking same parameters of lines (1) and (2) except ρ, we also computed the price for constant $\rho = -0.6$, and found the comparatively large fluctuation in the Monte Carlo method. Finally, we take 0.455 as the asymptotic price for $\rho = -0.6$ and get the price 0.468 from Eq.(38), which are not plotted in Fig.3. The averaged error for constant ρ is about 0.004. But in negative correlation region, since the true option price is comparatively small, the error of the Monte carlo method looks like a little large. Furthermore, we compute the option price with stochastic correlation. Since the procedure of proving the series solution Eq.(47) is rigorous and the absolute value of its fourth term is generally less that one basis point, Eq.(47) highlighting the first three terms could be treated as a benchmark for other numerical approaches. As has been expected, one more dimension, i.e., the Jacobi process, involved in the Monte carlo method, magnifies computational error. The price from the first three terms of Eq.(47) for $\rho_0 = 0.6$, $\bar{\rho} = 0.5$ plotted in line (3) is 0.586, but the asymptotic price from the Monte Carlo method plotted in line (4) is 0.580. Another little bigger difference takes places in $\rho_0 = 0.6$, $\bar{\rho} = -0.5$ case, where line (5) for Eq.(47) gives price 0.188. But in line (6), the Mont Carlo method always presents a comparatively large fluctuation, which raise the difficulty to determine the asymptotic price. Finally, we take 0.196 as its asymptotic value for $\rho_0 = 0.6$, $\bar{\rho} = -0.5$, whose relatively large error is due to the negative correlation induced by the negative equilibrium value during most of the life of the option. In the stochastic correlation cases, it seems that the averaged error of the Monte carlo method is about 0.007, which is almost double of the one in constant correlation case. It could be concluded that, using fast Fourier transformation, Eq.(47) is more reliable, efficient and accurate than the Monte carlo method.

FIG. 3. Lines 1 and 2 give a comparison between the correlation option price of Eq.(37) Model from Eq.(38) with constant correlation and the one from the Monte Carlo method. Lines (1) for Eq.(38) and (2) for the Monte Carlo method share same parameters, $K_1 = K_2 = 1$, $S_1(0) = 1.1$, $S_2(0) = 1.2$, $\tau = 5$, $r_d = 0.1$, $r_f = 0.05$, $\rho = 0.6$, $\bar{\rho}_1 = \bar{\rho}_2 = -0.2$, $Q_0 = 0.8$, $\sigma_1 = 0.5$, $\sigma_2 = 0.6$, $\theta_q = 0.5$, $\kappa_q = 1.0$, and $\sigma_q = 0.5$. Lines (3) and (4) are from Eq.(47) with random correlation and the Monte Carlo method, which share same parameters as in lines (1) and (2) except $\bar{\rho} = 0.5$, $\rho_0 = 0.6$, $h = -f = 0.8$, $g = 1.7$, $\sigma_3 = 1.0$. Lines (5) and (6) are from Eq.(47) with random correlation and the Monte Carlo method, which share same parameters as in lines (3) and (4) except $\bar{\rho} = -0.5$. In this Figure, horizontal coordinate, M, is the number of the simulated path, and vertical one is the price of correlation option, C.



5. CONCLUSION

Correlation risk in general sense is generally defined as the difference between implied and realized correlation for a given maturity, which can be frequently found in the market. To derive the pricing formula analytically, we assume the correlation risk premium is linearly proportional to the current level of correlation, which is used to introduce a kind of stochastic process, i.e., $d\rho = (\bar{\rho} - g\rho)dt + \sigma\sqrt{(h-\rho)(\rho-f)}dW$ to describe the random walk of correlation coefficients. In our scheme, the correlation coefficient wanders around the mean value within the region from the upper bound $h \leq 1$ to the lower one $f \geq -1$. Of course, there are many other ways to define correlation risk premium, and, thus, other risk neutral correlation processes are also possible, e.g., $d\rho = [\bar{\rho} + g \ln \sqrt{(h-\rho)(\rho-f)}]dt + \sigma\sqrt{(h-\rho)(\rho-f)}dW$, which could guarantee the correlation coefficient never hits the bounds. The pricing equation for stochastic correlation coefficient is derived. The obvious difference is numerically found as the quick diffusion happens in motions of correlation coefficients, which indicates that we have to use this model to eliminate correlation risk. At last, we tried to find whether a series solution for pricing the stochastic volatility and correlation simultaneously is available. Supposing that the stochastic process of correlation is independent, we finally developed a series solution for pricing both of volatility and correlation risks, i.e., Eq.(47) , whose second and third terms occupy three, five, or ten percents of total value, which could be used to capture the feature of structure of implied correlation. Consequently, we could use the trading strategy to hedge the correlation risk. For example, if trading a basket option on stocks highlighted in Standard and Poor's 500, to hedge the correlation risk involved, the trader could manipulate an option on the Standard and Poor index. An alternative way is that for instance, trading a basket option results in a correlation risk, but it can be hedged by applying a best-of option highlighting the component of the basket option. Namely, in the currency market, if the correlation risk results from an option on the basket consisting of U.S. dollar, Japanese Yen, and British pounds, then, a best-of call allows the owner to buy any of the currencies that increases the most at the maturity of the option, which can be used to hedge the risk.

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