

## Comparing Competitive Equilibria with Equilibria of Labor-Managed and Capital-Managed Economies in OLG Models\*

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This paper compares a private ownership (competitive) economy, a labor-managed economy and a capital-managed economy in an overlapping-generations framework. Under standard and rather weak assumptions, the sets of equilibria for the two first economies are identical, in line with a result already shown in a static setting. We also prove that the set of competitive equilibria are included in the set of equilibria of capital-managed economies, but the converse is not true. However, using some smoothness assumptions, we show that an equilibrium of a capital-managed economy may be a competitive equilibrium.

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### 1. INTRODUCTION

This paper compares in an overlapping-generations model (Allais [1947] - Diamond [1965] - Samuelson [1958]) a private ownership (competitive) economy, a labor-managed economy and a capital-managed economy.

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In a labor-managed economy individuals maximize value-added per worker; there is no labor market, but the capital market is competitive. In a capital-managed economy, on the other hand, individuals maximize the return from their savings in a non-cooperative manner; there is no capital market, but the labor market is competitive.

Whereas the concept of a capital-managed economy is to the best of our knowledge a theoretical novelty, there is a significant strand of literature comparing labor-managed with competitive economies (see the seminal work of Vanek [1970], Meade [1972] and Drèze [1976], [1989]). The key result of this literature - the equivalence between the equilibrium allocations - hinges upon a free-entry assumption. Absent this assumption, equivalence may not hold (see *e.g.*, Pestieau et Thisse [1979])<sup>1</sup>.

The equivalence result still holds in an OLG framework. This implies that a labor-managed economy can be Pareto-inefficient (due to a possible over-accumulation of capital).

As for the comparison of equilibria of competitive and capital-managed economies, we obtain three results. First, the set of competitive equilibria is included in the set of equilibria of capital-managed economies. Second, we present an example showing that the set of equilibria of capital-managed economies is not included in the set of competitive equilibria. Third, we establish sufficient conditions under which an equilibrium of a capital-managed economy is a competitive equilibrium<sup>2</sup>.

The paper is organized as follows. In the next section, we present the model economy. In section 3, we present the definitions of the three equilibria compared in the paper. Section 4 compares competitive equilibria and equilibria of labor-managed economies. Section 5 compares competitive equilibria and equilibria of capital-managed economies. Section 6 concludes the paper.

## 2. THE MODEL

### 2.1. Agents

We make use of an overlapping-generations model à la Allais [1947] - Samuelson [1958] -Diamond [1965]. In each date  $t \in \mathbb{N}$ ,  $N_t \in \mathbb{R}_+$  identical<sup>3</sup>

<sup>1</sup>In Drèze [1989], it is shown that the equivalence property breaks down when there are incomplete markets. Kihlstrom and Laffont [2002] also study the free entry equilibria of a labor-managed economy with incomplete contingent claims markets. They prove the existence of the equilibrium and show that it is Diamond inefficient. But they prove that introducing an insurance system internal to the firm restores Diamond efficiency. This is because the equilibrium is equivalent to an entrepreneurial stock market equilibrium.

<sup>2</sup>Production and utility functions must be differentiable; in addition, utility functions must satisfy the Inada conditions and be strictly quasi-concave.

<sup>3</sup>The results of this paper do not depend upon this assumption. We use it for simplicity.

agents are born. Life lasts two periods. The population growth rate  $n$  satisfies  $n > -1$ . Each agent works only during youth and supplies inelastically one unit of labor. The income of a young agent is denoted by  $\omega_t$ . This income finances both consumption ( $c_t$ ) and savings ( $s_t$ ). The returns to savings finance consumption in old age ( $d_{t+1}$ ).

In each date  $t$ , there exist three goods. The first good is labor; the second is a produced good which may be either saved or consumed and which is the numeraire; the third good is a capital good. Capital at date  $t$  is equal to the amount of the produced good of the previous period that has not been consumed. Capital depreciates fully within the production period.

The gross rate of return on savings is denoted with  $R_{t+1}$ . We shall only consider the case where both  $\omega_t$  and  $R_{t+1}$  are positive.

The budget constraints of an agent born at date  $t$  are:

$$c_t + s_t = \omega_t \quad (1)$$

$$d_{t+1} = R_{t+1}s_t. \quad (2)$$

Corresponding to these constraints, there is an intertemporal budget constraint:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = \omega_t. \quad (3)$$

The consumptions  $c_t$  and  $d_{t+1}$  are the solutions of the following problem:

$$\begin{aligned} \max_{(c_t, d_{t+1}) \in \mathbb{R}_+^2} & U(c_t, d_{t+1}) \\ c_t + \frac{d_{t+1}}{R_{t+1}} &= \omega_t. \end{aligned} \quad (4)$$

where  $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is a life-cycle utility function. We shall assume:

**(H1)**  $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  is *monotonic*<sup>4</sup>.

## 2.2. The firm

In each date, there exists a unique firm<sup>5</sup>. This firm produces a quantity  $Y_t$  of an homogenous good according to a neo-classical production function  $F(.,.)$  which uses capital ( $K_t$ ) and labor ( $L_t$ ) as inputs. Let  $C$  be a given convex cone<sup>6</sup> in  $\mathbb{R}_+^2$  which has at least one element with positive coordinates. We assume:

<sup>4</sup>I.e.  $c' \geq c$  and  $d' \geq d \Rightarrow U(c', d') \geq U(c, d)$ .

<sup>5</sup>The results of the paper do not depend upon the number of firms providing that they use the same production set.

<sup>6</sup>A set  $C$  is a cone if  $x \in C, \lambda \geq 0$ , implies  $\lambda x \in C$  (see Berge [1997], page 140).

(H2)  $F : C \rightarrow \mathbb{R}_+$  is concave, homogenous of degree one<sup>7</sup>, and monotonic. Moreover, when  $C$  is not open,  $F$  is continuous.

### 3. INTERTEMPORAL EQUILIBRIA OF THREE ECONOMIC SYSTEMS

In this section, we present three notions of equilibria for three economic systems. The initial capital stock  $\bar{K}_0$ , the amount of past consumption  $c_{-1}$ <sup>8</sup>, the initial level of population  $N_0$ , and the population growth rate  $n$  are given.

#### 3.1. Intertemporal competitive equilibrium

We first consider the traditional case of a private ownership economy. There are markets for all goods at all dates. Agents supply labor and demand a good which will be either consumed or invested. At all dates a firm supplies this good and hires labor and capital.

Let the wage rate be  $w_t$  and the gross return of capital be  $R_t$ .

**Definition 1:** An intertemporal competitive equilibrium is a sequence of vectors  $(R_t^*, w_t^*, c_t^*, d_t^*, s_t^*, K_t^*, L_t^*)_{t \geq 0}$  with positive coordinates such that for all  $t \geq 0$  :

- 1)  $U(c_t^*, d_{t+1}^*) \geq U(c_t, d_{t+1})$ , for all  $(c_t, d_{t+1})$  in  $\mathbb{R}_+^2$  such that  $c_t + \frac{d_{t+1}}{R_{t+1}^*} = w_t^*$ , and  $d_0^* = R_0^* \bar{K}_0 / N_0$ ;
- 2)  $F(K_t^*, L_t^*) - w_t^* L_t^* - R_t^* K_t^* = 0 \geq F(K_t, L_t) - w_t^* L_t - R_t^* K_t$ , for all  $(K_t, L_t)$  in  $C$ ;
- 3)  $L_t^* = N_t$ ;
- 4)  $K_{t+1}^* = N_t s_t^*$  with  $s_t^* = w_t^* - c_t^*$ , and  $K_0^* = \bar{K}_0$ ;
- 5)  $F(K_t^*, L_t^*) = N_t c_t^* + N_{t-1} d_t^* + K_{t+1}^*$ .

Hence, we define an intertemporal competitive equilibrium as a sequence of temporary equilibria with perfect foresights. The latter are the equilibria studied by Allais [1947], Samuelson [1958] and Diamond [1965].

The first condition of the definition corresponds to the maximization of the intertemporal utility function (notice here that  $\omega_t = w_t$ ). The second condition corresponds to profit maximization. The three last conditions are the equilibrium conditions on the markets for the production factors (labor and capital) and the market for the produced good.

#### 3.2. Intertemporal equilibrium of a labor-managed economy

<sup>7</sup>A function  $F$  is homogenous of degree one if:  $\forall \lambda > 0, \forall (K, L) \in C, F(\lambda K, \lambda L) = \lambda F(K, L)$ . Note that since  $(0, 0) \in C$ , under H2,  $F(0, 0) = 0$ .

<sup>8</sup>This is required in order to define the utility of old agents at date zero.

We shall now consider an economic system in which agents are member of a cooperative. The members of the cooperative choose their number so as to maximize their average income (the value-added per worker). In this economic system, capital is still hired in a competitive way.

Let  $V_t$  denotes the average income per worker and  $R_t$  the gross return on capital. The members of the cooperative solve the following problem:

$$\max_{(K_t, m_t) \in \{(K, L) : (K, L) \in C, L > 0\}} V_t = \frac{F(K_t, m_t) - R_t K_t}{m_t} \quad (5)$$

In equilibrium, the number of workers must be equal to  $N_t$  at any date  $t$  (this is because there is only one firm). When they receive the (maximal) average income, agents determine their savings which, in turn, determine next period's capital stock. This motivates the following definition:

**Definition 2:** *An intertemporal equilibrium of a labor-managed economy is a sequence of vectors  $(R_t^*, V_t^*, c_t^*, d_t^*, s_t^*, K_t^*, m_t^*)_{t \geq 0}$  with positive coordinates such that for all  $t \geq 0$  :*

- 1)  $U(c_t^*, d_{t+1}^*) \geq U(c_t, d_{t+1})$ , for all  $(c_t, d_{t+1})$  in  $\mathbb{R}_+^2$  such that  $c_t + \frac{d_{t+1}}{R_{t+1}^*} = V_t^*$ , and  $d_0^* = R_0^* \bar{K}_0 / N_0$ ;
- 2)  $V_t^* := \frac{F(K_t^*, m_t^*) - R_t^* K_t^*}{m_t^*} \geq \frac{F(K_t, m_t) - R_t^* K_t}{m_t}$ , for all  $(K_t, m_t) \in \{(K, L) : (K, L) \in C, L > 0\}$ ;
- 3)  $m_t^* = N_t$ ;
- 4)  $K_{t+1}^* = N_t s_t^*$  with  $s_t^* = V_t^* - c_t^*$ , and  $K_0^* = \bar{K}_0$ ;
- 5)  $F(K_t^*, m_t^*) = N_t c_t^* + N_{t-1} d_t^* + K_{t+1}^*$ .

The first equation still corresponds to the maximization of intertemporal utility. The second equation describes the maximization of the average income of the cooperative members (notice that here  $\omega_t = V_t$ ). The last two equations are the equilibrium conditions for the capital and the produced good markets.

### 3.3. Intertemporal equilibrium of a capital-managed economy

A partial equilibrium version of this equilibrium was first presented in Otani and El-Hodiri [1987] (page 199) who studied what they call a capital-managed firm.

We still assume that there is a unique firm. In each date  $t$ , the  $N_t$  young agents work (the labor market is competitive). Savings decisions are however made in a different context.

Consider indeed an agent born at date  $t$ . Let  $a_t^j$  be the (positive) amount of savings of the other  $N_t - 1$  agents. We define the returns from an agent's

savings as:

$$\frac{a_t}{a_t + a_t^j} \max \left\{ 0, F(a_t + a_t^j, L_{t+1}) - w_{t+1} L_{t+1} \right\} \quad (6)$$

where  $a_t \in [0, w_t]$  and  $(a_t + a_t^j, L_{t+1}) \in C$ . We will comment below the use of the max operator.

When  $(a_t + a_t^j, L_{t+1}) \in C$ , the budget constraints faced by an agent are:

$$c_t + a_t = w_t \quad (7)$$

$$d_{t+1} = \frac{a_t \max \left\{ 0, F(a_t + a_t^j, L_{t+1}) - w_{t+1} L_{t+1} \right\}}{a_t + a_t^j}. \quad (8)$$

In order to determine their savings, each agent solves:

$$\max_{a_t \in [0, w_t] \cap \{a : (a + a_t^j, L_{t+1}) \in C\}} U \left( w_t - a_t, \frac{a_t \max \left\{ 0, F(a_t + a_t^j, L_{t+1}) - w_{t+1} L_{t+1} \right\}}{a_t + a_t^j} \right) \quad (9)$$

The use of the operator max can now be justified. For some values of  $a_t$ , it may happen that the returns are negative. But the utility function is not defined in this case. The use of the max operator enables us to restrict the maximization problem to the subset of  $[0, w_t]$  for which the returns are non-negative.

**Definition 3:** *An intertemporal equilibrium for a capital-managed economy is a sequence of vectors  $(w_t^*, c_t^*, d_t^*, a_t^*, K_t^*, L_t^*)_{t \geq 0}$  with positive coordinates such that for all  $t \geq 0$ :*

- 1)  $U(w_t^* - a_t^*, \frac{a_t^* \max \{0, F(N_t a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*\}}{N_t a_t^*}) \geq U(w_t^* - a_t, \frac{a_t \max \{0, F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*\}}{a_t + (N_t - 1)a_t^*})$ ;  
for all  $a_t$  in  $[0, w_t] \cap \{a : (a_t + (N_t - 1)a_t^*, L_{t+1}^*) \in C\}$ ;  $N_0 d_0^* = F(\bar{K}_0, L_0^*) - w_0^* L_0^*$ ;
- 2)  $F(N_t a_{t-1}^*, L_t^*) - w_t^* L_t^* \geq F(N_t a_{t-1}^*, L_t) - w_t^* L_t$ , for all  $L_t$  in  $\mathbb{R}_+ \cap \{L : (N_t a_{t-1}^*, L) \in C\}$  (with  $N_0 a_{-1}^* = \bar{K}_0$ );
- 3)  $L_t^* = N_t$ ;
- 4)  $F(N_t a_{t-1}^*, L_t^*) = N_t c_t^* + N_{t-1} d_t^* + N_t a_t^*$  and  $F(\bar{K}_0, L_0^*) = N_0 c_0^* + N_{-1} d_0^* + N_0 a_0^*$  (for all  $t \geq 1$ ).

Notice that in equilibrium, savings is the outcome of a (symmetric) Nash equilibrium between agents. Finally, notice that here  $\omega_t = w_t^{a^*}$ .

**4. EQUIVALENCE OF COMPETITIVE AND LABOR MANAGED-ECONOMIES**

We state our first equivalence property.

**THEOREM 1.** *Assume H2. Then the set of intertemporal competitive equilibria and intertemporal equilibria of labor-managed economies are identical.*

*Proof.* We have to prove that the sequence:

$$(R_t^*, w_t^*, c_t^*, d_t^*, s_t^*, K_t^*, L_t^*)_{t \geq 0}$$

is an intertemporal competitive equilibrium if and only if the sequence

$$(R_t^*, V_t^*, c_t^*, d_t^*, s_t^*, K_t^*, m_t^*)_{t \geq 0} \equiv (R_t^*, w_t^*, c_t^*, d_t^*, s_t^*, K_t^*, L_t^*)_{t \geq 0}$$

is an intertemporal equilibrium of a labor-managed economy.

( $\Rightarrow$ ) By assumption,  $(R_t^*, w_t^*, c_t^*, d_t^*, s_t^*, K_t^*, L_t^*)_{t \geq 0}$  is an intertemporal competitive equilibrium. From condition 2 of definition 1, profit is maximized:  $\forall (K_t, L_t) \in C$ ,

$$F(K_t^*, L_t^*) - w_t^* L_t^* - R_t^* K_t^* = 0 \geq F(K_t, L_t) - w_t^* L_t - R_t^* K_t. \quad (10)$$

From this, and since by assumption  $L_t^* > 0$ , it follows that:

$$V_t^* := \frac{F(K_t^*, L_t^*) - R_t^* K_t^*}{L_t^*} = w_t^* \geq \frac{F(K_t, L_t) - R_t^* K_t}{L_t} \quad (11)$$

$$\forall (K_t, L_t) \in \{(K, L) : (K, L) \in C, L > 0\}$$

Let  $m_t^* = L_t^*$ . Then, an intertemporal competitive equilibrium is an intertemporal equilibrium of a labor-managed economy since all the conditions of definition 2 are satisfied.

( $\Leftarrow$ ) Conversely, assume that the sequence  $(R_t^*, V_t^*, c_t^*, d_t^*, s_t^*, K_t^*, m_t^*)_{t \geq 0}$  is an intertemporal equilibrium of a labor-managed economy. By assumption, from condition 2 of definition 2, the income per laborer is maximized:  $\forall (K_t, m_t) \in \{(K, L) : (K, L) \in C, L > 0\}$

$$V_t^* := \frac{F(K_t^*, m_t^*) - R_t^* K_t^*}{m_t^*} \geq \frac{F(K_t, m_t) - R_t^* K_t}{m_t}, \quad (12)$$

From this, it follows that :  $\forall (K_t, m_t) \in \{(K, L) : (K, L) \in C, L > 0\}$

$$F(K_t^*, m_t^*) - V_t^* m_t^* - R_t^* K_t^* \geq F(K_t, m_t) - V_t^* L_t - R_t^* K_t. \quad (13)$$

On the other hand, from H2, when  $(K_t, m_t) = (0, 0)$ ,  $F(K_t, m_t) - V_t^* L_t - R_t^* K_t = 0$ . It remains to show that the preceding inequality is true for the possible remaining parts of  $C$ . Indeed, if  $C$  is not open, there may exit vectors in  $C$  whose second coordinate  $L$  is nil. But since  $C$  is convex, these points  $(K, 0)$  can always be reached from a line segment included in  $C$  and such that all points of the segment but  $(K, 0)$  have a positive second coordinate. By H2,  $F(., .)$  is continuous. Using the inequality above, this implies that  $F(K_t, 0) - R_t^* K_t \leq 0$ .

Finally, let us posit  $w_t^* := V_t^*$ . It is easy to see that all the conditions of definition 1 (of an intertemporal competitive equilibrium) are satisfied. ■

In a static framework, this equivalence property is well-known (see Drèze [1976], [1989], Vanek [1970]). It is often interpreted as follows. An economy with labor-management is as efficient as a “competitive” economy. We have extended this result to a dynamic setting. In this framework, it is well-known that competitive equilibria may be Pareto-inefficient (i.e., when there is over-accumulation of capital (see *e.g.* Diamond [1965])). It follows that an economy with labor-management could also be Pareto-inefficient.

## 5. EQUIVALENCE OF COMPETITIVE EQUILIBRIA AND CAPITAL-MANAGED ECONOMIES

### 5.1. Intertemporal competitive equilibria are intertemporal equilibria of capital-managed economies

We first state that an intertemporal competitive equilibrium is an intertemporal equilibrium of a capital-managed economy.

**PROPOSITION 1.** *Assume H1 and H2. Then the set of intertemporal competitive equilibria is a subset of the intertemporal equilibria of a capital-managed economy.*

*Proof.* Let an intertemporal competitive equilibrium be given. Condition 2 of definition 1 (of a competitive equilibrium) may be written:  $\forall a_t \in [0, w_t]$ ,  $\forall L_{t+1} \in \mathbb{R}_+$ , such that  $(a_t^i + (N_t - 1)s_t^*, L_{t+1}) \in C$

$$\begin{aligned} & F(K_{t+1}^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^* - R_{t+1}^* K_{t+1}^* = 0 \\ & \geq F(a_t^i + (N_t - 1)s_t^*, L_{t+1}) - w_{t+1}^* L_{t+1} - R_{t+1}^* (a_t^i + (N_t - 1)s_t^*). \end{aligned} \quad (14)$$



One has:

$$U(w_t^* - a_t, \frac{a_t \max \{0, F(a_t + (N_t - 1)s_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*\}}{a_t + (N_t - 1)s_t^*}) \tag{15}$$

$$= U(w_t^* - a_t, \frac{a_t \max \{0, (F(a_t + (N_t - 1)s_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^* \pm R_{t+1}^*(a_t + (N_t - 1)s_t^*))\}}{a_t + (N_t - 1)s_t^*})$$

$$\leq U(w_t^* - a_t, \frac{a_t R_{t+1}^*(a_t + (N_t - 1)s_t^*)}{a_t + (N_t - 1)s_t^*}) \tag{16}$$

$$= U(w_t^* - a_t, a_t R_{t+1}^*). \tag{17}$$

Notice that the third inequality stems from the monotonicity assumption in H1 and profit maximization.

From condition 1 of definition 1 (maximization of intertemporal utility), one has:

$$U(w_t^* - a_t, a_t R_{t+1}^*) \leq U(w_t^* - s_t^*, s_t^* R_{t+1}^*). \tag{18}$$

From the preceding inequalities, one gets:  $\forall a_t \in [0, w_t]$ , such that  $(a_t + (N_t - 1)s_t^*, L_{t+1}^*) \in C$ ,

$$U(w_t^* - a_t, \frac{a_t \max \{0, (F(a_t + (N_t - 1)s_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*)\}}{a_t + (N_t - 1)s_t^*}) \leq U(w_t^* - s_t^*, s_t^* R_{t+1}^*). \tag{19}$$

This is condition 1) of definition 3) with  $a_t^* = s_t^*$ . One concludes that when all agents “plays”  $s_t^*$ ,  $t \geq 0$ , one has a Nash equilibrium. The remaining equations of the definition of an intertemporal competitive equilibrium insure that the latter is an intertemporal equilibrium of a capital-managed economy. This establishes the proposition. ■

**5.2. A counterexample**

One can construct a counterexample showing that the converse of proposition 1 is generally false. Unfortunately, the details of the argument are cumbersome and have been relegated in the appendix. However, the gist of the argument can be seen in figure 1.

In figure 1, the intertemporal equilibrium of the capital-managed economy considered in the example is located at point A (more precisely, point A represents the couple of consumptions  $(c, d)$  chosen by any agent at such an equilibrium). The set of possible consumptions for this equilibrium is the area under the curve  $g(c)$ .

This equilibrium is not an intertemporal competitive equilibrium. Indeed, in a competitive equilibrium, the set of possible consumptions is the

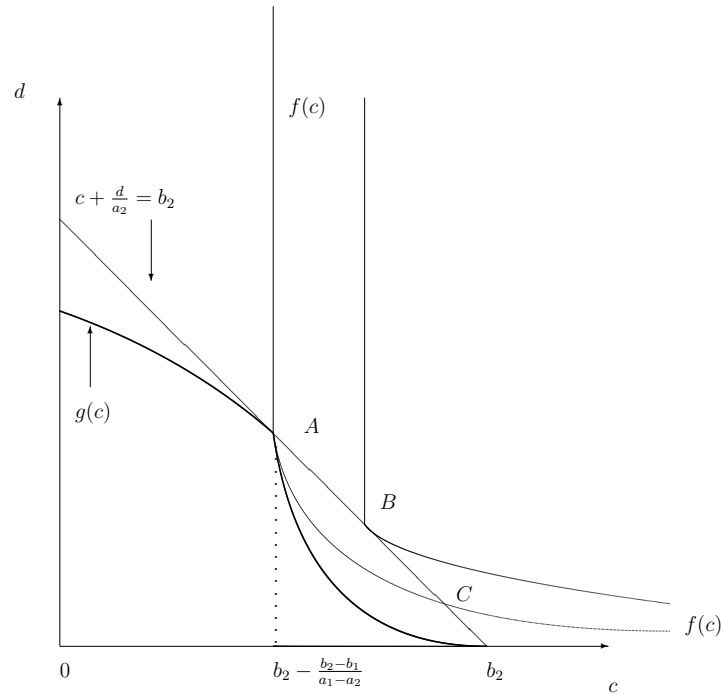


FIG. 1.

area under the line corresponding to the intertemporal budget constraint  $c + d/a_2 = b_2$  (the rate of return on savings is taken as given and is equal to  $a_2$ ;  $b_2$  is the value of the wage rate). The indifference curve corresponding to the utility reached at the intertemporal equilibrium of a capital-managed economy is described by the curve  $f(c)$ . This curve intersects the intertemporal budget line at point  $C$ . Clearly, by choosing consumptions corresponding to point  $B$ , an agent could get a strictly higher intertemporal utility. This proves that the equilibrium of the capital-managed economy considered is not a competitive equilibrium.

### 5.3. A sufficient condition for an intertemporal equilibrium of a capital-managed economy to be an intertemporal competitive equilibrium

In what follows, we shall get an equivalence property under some smoothness assumptions.

**(H3)** In addition to H1 and H2, we assume that both  $U(.,.)$  and  $F(.,.)$  are continuously differentiable on the interior of their domains. Moreover  $U(.,.)$  is strictly quasiconcave<sup>9</sup> and satisfies the Inada conditions<sup>10</sup>.

**THEOREM 2.** Assume H3 and that for all  $t \geq 0$   $(K_t, N_t)$  is in the interior of  $C$ . Then an intertemporal equilibrium of a capital-managed economy is an intertemporal competitive equilibrium.

*Proof.* Let an intertemporal equilibrium of a capital-managed economy be given. Consider any date  $t$ . By assumption, the function:

$$U(w_t^* - a_t, \frac{a_t \max \{0, F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*\}}{a_t + (N_t - 1)a_t^*}), \tag{20}$$

defined for all  $a_t$  in  $[0, w_t] \cap \{a : (a + (N_t - 1)a_t^*, L_{t+1}^*) \in C\}$  reached a maximum at  $a_t^*$  which is in  $]0, w_t[$  and such that  $(N_t a_t^*, N_{t+1})$  is in the interior of  $C$ . Hence, the following first-order optimality condition holds true at  $a_t = a_t^*$ :

$$\begin{aligned} & U'_c(w_t^* - a_t, \frac{a_t(F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*)}{a_t + (N_t - 1)a_t^*}) \\ = & (\frac{((N_t - 1)a_t^*)(F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*)}{(a_t + (N_t - 1)a_t^*)^2} + \frac{a_t F_1(a_t + (N_t - 1)a_t^*, L_{t+1}^*)}{a_t + (N_t - 1)a_t^*}) \\ & \times U'_d(w_t^* - a_t, \frac{a_t(F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^*)}{a_t + (N_t - 1)a_t^*}). \end{aligned} \tag{21}$$

But by homogeneity of degree 1 of the production function and the optimality of labor choice by the firm, at  $a_t = a_t^*$  one has:

$$\begin{aligned} & F(a_t + (N_t - 1)a_t^*, L_{t+1}^*) - w_{t+1}^* L_{t+1}^* \\ = & F_1(a_t + (N_t - 1)a_t^*, L_{t+1}^*)(a_t + (N_t - 1)a_t^*) \\ \equiv & R_{t+1}^*(a_t + (N_t - 1)a_t^*). \end{aligned} \tag{22}$$

Hence, the optimality condition reduces to:

$$U'_c(w_t^* - a_t^*, a_t^* R_{t+1}^*) = R_{t+1}^* U'_d(w_t^* - a_t^*, a_t^* R_{t+1}^*). \tag{23}$$

<sup>9</sup>We stick to the usual definition of quasi-concavity: a real-valued function  $f$  defined in a convex set  $C \subset \mathbb{R}^n$  is quasi-concave, if for all  $\alpha$  in  $\mathbb{R}$ , the set  $\{x : f(x) \geq \alpha\}$  is convex (whenever it is non-empty). A real-valued function  $f$  is strictly quasi-concave on a convex set  $C$  if,  $f(x) \geq t$  and  $f(x') \geq t$  implies  $f(\lambda x + (1 - \lambda)x') > t$ , for all  $t \in \mathbb{R}$ ,  $x \neq x'$ ,  $\lambda \in ]0, 1[$ .

<sup>10</sup>The Inada conditions are as follows: for all  $d > 0$ ,  $\lim_{c \rightarrow 0} \frac{\partial U}{\partial c}(c, d) = +\infty$ , and for all  $c > 0$ ,  $\lim_{d \rightarrow 0} \frac{\partial U}{\partial d}(c, d) = +\infty$ .

Now, we show that an agent facing the problem:

$$\max_{s \in [0, w_t^*]} U(w_t^* - s, sR_{t+1}^*) \quad (24)$$

would choose  $s = a_t^*$ .

Suppose the contrary. Let  $s$  be the solution to the above problem with  $s \neq a_t^*$ . Since  $U(., .)$  satisfies the Inada conditions, necessarily:

$$U'_c(w_t^* - s, sR_{t+1}^*) = R_{t+1}^* U'_d(w_t^* - s, sR_{t+1}^*). \quad (25)$$

By Theorem M.C.3. of Mas-Colell, Whinston and Green [1995] (page 934), since  $U(., .)$  is strictly quasiconcave, and since under the Inada assumption  $\nabla U \neq 0$  on  $\mathbb{R}_{++}^2$ ,  $U(w_t^* - s, sR_{t+1}^*) > U(w_t^* - a_t^*, a_t^* R_{t+1}^*)$  implies:

$$\langle \nabla U(w_t^* - a_t^*, a_t^* R_{t+1}^*), (a_t^* - s, R_{t+1}^*(s - a_t^*)) \rangle > 0. \quad (26)$$

But since  $\nabla U(w_t^* - a_t^*, a_t^* R_{t+1}^*) = U'_d(w_t^* - a_t^*, a_t^* R_{t+1}^*) \times (R_{t+1}^*, 1)$  this is impossible. We conclude that the savings choices of agents satisfy:  $s = a_t^*$ .

The remaining conditions of definition 1 are easily shown to be satisfied. ■

Why is it that in the smoothness case an equilibrium of a capital-managed economy is an intertemporal competitive equilibrium? There are four features that yield the result and which are instrumental in the proof. First of all, the Inada conditions ensure that one will get an interior solution whatever the equilibrium notion. Second, since the production function is smooth, due to the Euler's Theorem and the fact that the labor market is competitive, the return of savings is the marginal productivity of capital, even at an equilibrium of a capital-managed economy. Third, due to the smoothness of the utility function, the standard condition forbidding further intertemporal tradeoffs holds true (there is no beneficial intertemporal arbitrage). Fourth, by strict quasiconcavity of the utility function, this marginal condition is sufficient for optimality.

## 6. CONCLUSION

This paper was devoted to the comparison in a dynamic setting of three different economic equilibria. We confirm the result already obtained in a static framework that the sets of competitive equilibria and equilibria with labor-management are identical. Such an equivalence property does not hold in general for the sets of competitive equilibria and equilibria of capital-managed economies. The former is included in the latter, but the converse is false as a counterexample shows. However, under some smoothness assumptions, an equilibrium of a capital-managed economy may be

a competitive equilibrium. Given the approach followed in this paper, a natural extension of the present research is to test the robustness of the result by enriching the assumptions regarding preferences. The analysis could be extended to the case of intergenerational altruism.

APPENDIX

In this appendix we provide an example that an equilibrium of a capital-managed economy is not a competitive equilibrium.

The example relies on the following assumptions:

(H) (*Assumptions of the example*)

- a)  $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+, (c, d) \mapsto \min \left\{ c, (cd)^{\frac{1}{2}} \right\};$
- b)  $F : \mathbb{R}_{++}^2 \cup \{(0, 0)\} \rightarrow \mathbb{R}_+,$

$$F(K, L) = \begin{cases} a_1K + b_1L & \text{if } \frac{K}{L} < \frac{b_2-b_1}{a_1-a_2}, L > 0 \\ a_2K + b_2L & \text{if } K = \frac{b_2-b_1}{a_1-a_2}L \\ (a_2\frac{b_2-b_1}{a_1-a_2} + b_2)L & \text{if } \frac{K}{L} > \frac{b_2-b_1}{a_1-a_2}, L > 0 \end{cases}$$

where:  $a_1 > a_2 > 0, b_2 > b_1 > 0, b_2 = \frac{1+a_2}{1+2a_2-a_1}b_1$  and:  $1 > a_2, 2a_2 - a_1 > 0;$

- c) *The size of each generation is equal to 2;*
- d)  $K_0 = 2\frac{b_2-b_1}{a_1-a_2}.$

LEMMA 1. *The production function in H b) is monotonic, concave, homogenous of degree one and continuous on the cone  $\mathbb{R}_{++}^2 \cup \{(0, 0)\}$*

*Proof.* The production function is clearly homogenous of degree one. Furthermore, it is quasiconcave. To see this, take  $(K_1, L_1)$  and  $(K_2, L_2)$  in the domain of  $F$  such that  $F(K_1, L_1) \geq \alpha$  and  $F(K_2, L_2) \geq \alpha$ , where  $\alpha$  is a real number. We need to show that for all  $\lambda$  in  $[0, 1], F(\lambda K_1 + (1 - \lambda)K_2, \lambda L_1 + (1 - \lambda)L_2) \geq \alpha.$

This is easily seen when either  $K_1/L_1$  and  $K_2/L_2$  are lower than  $\frac{b_2-b_1}{a_1-a_2}$  or when both are higher than  $\frac{b_2-b_1}{a_1-a_2}$  (and  $L_1$  and  $L_2$  positive). Also, when either  $(K_1, L_1)$  or  $(K_2, L_2)$  are equal to  $(0, 0)$  the result follows since  $F(., .)$  is homogenous of degree one and because, necessarily,  $\alpha \leq 0.$

Suppose now without loss of generality that  $K_1/L_1 \leq \frac{b_2-b_1}{a_1-a_2}$  and  $K_2/L_2 \geq \frac{b_2-b_1}{a_1-a_2}$  (with at least one inequality being strict and both  $L_1$  and  $L_2$  positive).

By assumption, we have:  $\alpha \leq a_1K_1 + b_1L_1.$  Hence, as  $L_1(a_1(K_1/L_1) + b_1) \leq L_1(a_1\frac{b_2-b_1}{a_1-a_2} + b_1), L_1(a_1\frac{b_2-b_1}{a_1-a_2} + b_1) \geq \alpha.$  We also have  $\alpha \leq$

$L_2(a_2 \frac{b_2-b_1}{a_1-a_2} + b_2) = L_2(a_1 \frac{b_2-b_1}{a_1-a_2} + b_1)$ . So when  $(\lambda K_1 + (1-\lambda)K_2)/(\lambda L_1 + (1-\lambda)L_2) \geq \frac{b_2-b_1}{a_1-a_2}$  one has :  $(\lambda L_1 + (1-\lambda)L_2)(a_2 \frac{b_2-b_1}{a_1-a_2} + b_2) \geq \alpha$ .

Suppose that  $(\lambda K_1 + (1-\lambda)K_2)/(\lambda L_1 + (1-\lambda)L_2) \leq \frac{b_2-b_1}{a_1-a_2}$ . Since  $K_2/L_2 \geq \frac{b_2-b_1}{a_1-a_2}$ , one has  $\alpha \leq L_2(a_2 \frac{b_2-b_1}{a_1-a_2} + b_2) = L_2(a_1 \frac{b_2-b_1}{a_1-a_2} + b_1) \leq a_1 K_2 + b_1 L_2$ . From this, one easily sees that  $F(.,.)$  is quasi-concave (since  $\alpha \leq a_1 K_1 + b_1 L_1$ ).

Let us see now that  $F$  is concave. Clearly,  $F$  is strictly positive on  $\mathbb{R}_{++}^2$ . Hence, from Berge [1997], Theorem 3, page 208,  $F(.,.)$  is concave (Berge assumes:  $F(x) > 0$  implies  $F(\lambda x) = \lambda x$  whenever  $\lambda > 0$ , a requirement which holds true in our example).

As for the continuity property, we know that a concave function is continuous on the interior of its domain. Hence, we only have to check that it is continuous at  $(0, 0)$ . This is the case, since  $F$  is majorized by the function  $\max \left\{ a_1 K + b_1 L, a_2 K + b_2 L, L(a_2 \frac{b_2-b_1}{a_1-a_2} + b_2) \right\}$  which is clearly continuous at  $(0, 0)$ .

Finally, it is not difficult to show that  $F(.,.)$  is monotonic. ■

Notice that the utility function is quasiconcave and monotonic. Hence, according to the preceding Lemma both assumptions H1) and H2) are satisfied.

In what follows, we shall first prove that the sequence with all terms equal to

$$(w_t^*, c_t^*, d_t^*, a_t^*, K_t^*, L_t^*) = (b_2, b_2 - \frac{b_2 - b_1}{a_1 - a_2}, b_2 - \frac{b_2 - b_1}{a_1 - a_2}, \frac{b_2 - b_1}{a_1 - a_2}, \frac{2(b_2 - b_1)}{a_1 - a_2}, 2)$$

is an intertemporal equilibrium of a capital-managed economy. Second, we shall prove that this equilibrium is not competitive.

**1)  $(w_t^*, c_t^*, d_t^*, a_t^*, K_t^*, L_t^*)$  is an intertemporal equilibrium of a capital-managed economy**

To show this, we need to prove that choosing a savings rate  $s_t^* = (b_2 - b_1)/(a_1 - a_2)$  for all  $t$  is optimal for any agent. We also need to prove that when the wage rate  $w_t^* = b_2$ , choosing  $L_t = 2$  is indeed profit maximizing.

This last point is shown as follows. Notice that whenever  $L_t = 2$ , and  $K_t = 2(b_2 - b_1)/(a_1 - a_2)$  for all  $t \geq 0$ , the profit writes<sup>11</sup>  $\pi(2 \frac{b_2-b_1}{a_1-a_2}, 2) \equiv 2a_2 \frac{b_2-b_1}{a_1-a_2}$ . If  $L_t \geq 2$ ,  $\pi(2 \frac{b_2-b_1}{a_1-a_2}, L_t) = a_1 2 \frac{b_2-b_1}{a_1-a_2} + b_1 L_t - b_2 L_t$  which is always lower than  $\pi(2 \frac{b_2-b_1}{a_1-a_2}, 2)$ . Finally, when  $L_t \leq 2$ ,  $\pi(2 \frac{b_2-b_1}{a_1-a_2}, L_t) = L_t(a_2 \frac{b_2-b_1}{a_1-a_2} + b_2) - b_2 L_t = L_t a_2 \frac{b_2-b_1}{a_1-a_2}$  which is always lower than  $2a_2 \frac{b_2-b_1}{a_1-a_2}$ .

Let us go back to the first point. We have to show that  $s_t^*$  is optimal from the agents point of view.

<sup>11</sup>We note  $\pi(.,.)$  the value of the profit as a function of capital and labor.

When  $L_t = 2$  for all  $t$ , the profit writes<sup>12</sup>:

$$\pi(K, 2) = \begin{cases} a_1 K + 2(b_1 - b_2) & \text{if } K \leq 2 \frac{b_2 - b_1}{a_1 - a_2} \\ a_2 K & \text{if } K = 2 \frac{b_2 - b_1}{a_1 - a_2} \\ 2a_2 \frac{b_1 - b_2}{a_1 - a_2} & \text{if } K \geq 2 \frac{b_2 - b_1}{a_1 - a_2} \end{cases}$$

As we look for a symmetric equilibrium, we shall first study the returns from savings for an agent when the other saves  $\frac{b_2 - b_1}{a_1 - a_2}$ . The returns to savings write:  $\varphi : [0, b_2] \rightarrow \mathbb{R}_+$ , is  $s \mapsto \frac{s}{s + s_t^*} \max \{0, \pi(s + s_t^*, 2)\}$ .

- $s \in \left[0, \frac{b_2 - b_1}{a_1 - a_2}\right]$

Notice that when  $K \leq 2 \frac{b_2 - b_1}{a_1 - a_2}$ ,  $\pi(K, 2) \geq 0$  if and only if  $K \geq \frac{2(b_2 - b_1)}{a_1}$ . Under our assumptions,

$$\frac{b_2 - b_1}{a_1 - 2} > 2 \frac{b_2 - b_1}{a_1} \Rightarrow \pi(s + s_t^*, 2) > 0, \forall s \in \left[0, \frac{b_2 - b_1}{a_1 - 2}\right] \tag{A.1}$$

Hence  $\varphi(s) > 0$ , when  $s \in \left[0, \frac{b_2 - b_1}{a_1 - 2}\right]$  and it is increasing.

One has:

$$\begin{aligned} \varphi(s) &= \frac{s}{s + s_t^*} (a_1(s + s_t^*) + 2(b_1 - b_2)) \\ &= a_1 s + \frac{2s(b_1 - b_2)}{s + s_t^*} \end{aligned} \tag{A.2}$$

Since  $b_1 < b_2$ ,  $\varphi(\cdot)$  is a convex function (it is the sum of two convex functions).

- When  $s = \frac{b_2 - b_1}{a_1 - a_2}$ ,  $\varphi(s) = a_2 \frac{b_2 - b_1}{a_1 - a_2}$ .

- When  $s \in \left[\frac{b_2 - b_1}{a_1 - a_2}, b_2\right]$ ,  $\varphi(s) = \frac{s 2a_2 \frac{b_2 - b_1}{a_1 - a_2}}{s + s_t^*}$  and  $\varphi(\cdot)$  is an increasing concave function.

Now, we see that  $s_t^*$  is a solution to:  $b_2 - s_t^* = a_2 s_t^*$  ( $c_t^* = d_{t+1}^*$ ). Given the feature of the utility function, the choice of savings is optimal if and only if there is no lower value  $s$  that yields the same utility level than  $s_t^*$ . Geometrically, this amounts to say that there is no  $(c, d)$  for which the indifference curve is tangent to the budget curve (*i.e.* the graph of  $\varphi(b_2 - c)$ ) (see figure 1).

The equation of the indifference curve is  $d = f(c) = \frac{(\frac{a_2}{1+a_2})^2 b_2^2}{c}$  (where we have use our assumptions regarding  $b_2$  and  $b_1$ )<sup>13</sup>. The equation of the

<sup>12</sup>When  $t = 0$ , we assume that the capital stock is equal to  $2 \frac{b_2 - b_1}{a_1 - a_2}$  and is equally shared by the two old agents. Each of them will then receive an income equal to  $a_2 \frac{b_2 - b_1}{a_1 - a_2}$ .

<sup>13</sup>That is  $\frac{a_2 b_1}{1+2a_2-b_1} = \frac{a_2 b_2}{1+a_2}$ .

budget curve is given by  $d = g(c) = \varphi(b_2 - c) = a_1(b_2 - c) + \frac{2(b_2 - c)}{b_2 - c + s_t^*}(b_1 - b_2)$ .

We take  $\left[b_2 - \frac{b_2 - b_1}{a_1 - a_2}, b_2\right]$  as the domain of  $f(\cdot)$  and  $g(\cdot)$ . One sees that  $h(c) = f(c) - g(c) = 0$  when  $c = b_2 - \frac{b_2 - b_1}{a_1 - a_2}$ .

A sufficient condition for  $c = b_2 - \frac{b_2 - b_1}{a_1 - a_2}$  to be optimal is that  $h'(c) > 0$  on  $\left]b_2 - \frac{b_2 - b_1}{a_1 - a_2}, b_2\right[$ , that is:

$$h'(c) = -\frac{\left(\frac{a_2}{1+a_2}\right)^2 b_2^2}{c^2} + a_1 + \frac{2s_t^*}{(b_2 - c + s_t^*)^2}(b_1 - b_2) > 0 \quad (\text{A.3})$$

But,  $h'(c) = f'(c) + \varphi'(b_2 - c)$ . On the domain of  $h'(\cdot)$ , one has  $f'(c) = -\frac{\left(\frac{a_2}{1+a_2}\right)^2 b_2^2}{c^2} > -1$ . So a sufficient condition for  $h'(c) > 0$ , is that  $\varphi'(b_2 - c) \geq 1$ . Since  $\varphi'(b_2 - c)$  is decreasing on the domain of  $h'(c)$ , a sufficient condition for  $h'(c) > 0$  is then  $\varphi'(0) > 1$ . The latter condition reduces to  $2a - a_1 > 0$  which is satisfied by assumption.

The reasoning above show that  $s_t^*$  is the optimal amount of savings for all  $t$ , and since  $w_t = b_2$  at each date, we have indeed found an intertemporal equilibrium of a capital-managed economy.

## 2) $(w_t^*, c_t^*, d_t^*, a_t^*, K_t^*, L_t^*)$ is not a competitive equilibrium

To show this, it is sufficient to prove that when the competitive rate of return on savings is  $a_2$ , the optimal choice is no more  $s_t^*$  ( $t$  being arbitrary)<sup>14</sup>. But notice that the indifference curve is strictly convex when  $c \geq b_2 - s_t^*$  (see also figure 1). Hence, it suffices to show that there is another affordable  $s$ , *i.e.* such that  $d = a_2 s$  and  $c = b_2 - s$ , and  $U(c, d) = U(b_2 - s_t^*, a_2 s_t^*)$ . This is achieved if there is a non-negative solution  $s$  to the equation  $j(s) = s^2 - s b_2 + \frac{U(b_2 - s_t^*, a_2 s_t^*)^2}{a_2} = 0$  on  $[0, s_t^*]$ . A sufficient condition for this is  $j'(s_t^*) > 0$ , and this obtains when  $1 > a_2$  which we have assumed<sup>15</sup>.

## REFERENCES

- Allais, Maurice, 1947. *Economie et Intérêt*. Imprimerie Nationale, Paris.
- Berge, Claude, 1997. *Topological Spaces*. Dover Editions.
- Diamond, Peter, 1965. National debt in a neo-classical growth model. *American Economic Review* **55**, 1126-1150.
- Drèze, Jacques-Henri, 1976. Some theory of labor management and participation. *Econometrica* **44**, 1125-1139.
- Drèze, Jacques-Henri, 1989. *Labour Management, Contracts And Capital Markets*, Basil Blackwell, Oxford, Royaume-Uni.

<sup>14</sup>Recall that  $\varphi(s_t^*) = a_2 s_t^*$  for all  $t$ .

<sup>15</sup>Notice that  $j'(s_t^*) = 2\frac{b_2 - b_1}{a_1 - a_2} - b_2 = (1 - a_2)\frac{b_2 - b_1}{a_1 - a_2}$ .



- Kihlstrom, Richard and Jean-Jacques Laffont, 2002. General equilibrium in a labor-managed economy with uncertainty and incomplete markets. *Annals of Economics and Finance* **3**, 185-217.
- Mas-Colell, Andreu, Whinston, Michael, and Jerry Green, 1995. *Microeconomic Theory*. Oxford Student Edition.
- Meade, James, 1972. The theory of labour-managed firms and of profit-sharing. *Economic Journal* **82**, 402-428.
- Otani, Y. and Mohamed El-hodiri, 1987. *Micro-Economic Theory*. Springer-Verlag.
- Pestieau, Pierre and Jacques-Francois Thisse, 1979. On imperfections and labor-management. *Economics Letters* **Vol. 3**, 353-356.
- Samuelson, Paul-Anthony, 1958. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy* **66**, 467-482.
- Vanek, J., 1970. *The General Theory Of Labor-Managed Market Economies*. Cornell University Press, Ithaca.