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Existence of an Equilibrium in a General Competitive Exchange Economy with Indivisible Goods and Money *

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We introduce a general competitive exchange economy with indivisible goods and money. There are finitely many agents and finitely many indivisible goods. Each agent is initially endowed with several units of each indivisible good and certain amount of money. Money is treated as a perfectly divisible good. The agents' preferences depend on the bundle of indivisible goods and the quantity of money they consume. Preferences are quite general and are not required to be quasilinear in money. We derive a necessary and sufficient condition for the existence of a competitive equilibrium in the economy. The model and the existence results we provide here unify and extend most models and existence results that have appeared in the literature.

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1. INTRODUCTION

Both the convexity assumption and the perfect divisibility assumption are two fundamental assumptions used in major modern economic analyses. Unfortunately, neither the convexity of production sectors nor the perfect divisibility of commodities is a convincingly realistic description of economic reality. As a matter of fact, the prominent feature of economic reality is the presence of large indivisibilities. How to deal with economies in the presence of indivisibilities has long been one of the major challenges in economic theory; see Lerner (1944), Debreu (1959), and Scarf (1986, 1994).

The aim of this paper is to establish equilibrium existence theorems for a general exchange economy in which agents exchange several indivisible goods and money. It is well known that competitive equilibrium theory runs into difficulties when indivisible goods are considered, or more generally, the quntities of goods are restricted to integers. Nevertheless, it has been shown by Quinzii (1984), Gale (1984), Svensson (1984), and Kaneko and Yamamoto (1986) that if there is a single divisible good (say money) in an economy and if each agent has utility for one indivisible object only, then there still exists an equilibrium under certain reasonable conditions. Their models extend the model of Shapley and Scarf (1974) in which no divisible goods are present. More general results have recently been obtained by e.g., Bikhchandani and Mamer (1997), Laan, Talman and Yang (1997), Bevia, Quinzii and Silva (1999), Gul and Stacchett (1999), Yang (2000), Danilov, Koshevoy, and Murota (2001), Murota and Tamura (2001). In these new models, agents can sell and buy several indivisible goods. It might also be worth pointing out that the models proposed in Bikhchandani and Mamer (1997), Bevia et al. (1999), Gul and Stacchett (1999), Murota and Tamura (2001) deal with the cases in which preferences are quasilinear in money, whereas those proposed in Quinzii (1984), Gale (1984), Svensson (1984), Keneko and Yamamoto (1986), Laan et al. (1997), and Danilov et al. (2001) deal with somehow more general situations where quasilinearity is not required.

In this paper we will introduce a more general competitive exchange economy with indivisible goods and money. There are finitely many agents and finitely many indivisible goods. Each agent is initially endowed with several units of each indivisible good and certain amount of money. Money is treated as a perfectly divisible good. The agents' preferences depend on the bundle of the quantity of indivisible goods and the quantity of money they consume. Preferences are quite general and are not required to be quasilinear in money. We derive necessary and sufficient conditions for the existence of a competitive equilibrium in the economy. It is further shown that our sufficient conditions are general and interesting enough to cover most existence conditions that have been introduced in the literature. Furthermore, this paper, to our knowledge, is also the first to present a necessary and sufficient condition for the existence of equilibrium in an economy where preferences can be quite general and quasilinearity is not required, whereas previous necessary and sufficient conditions established by Bikhchandani and Mamer (1997), and Murota and Tamura (2001) can only apply to the case in which agents have quasi-linear utilities in money.

The rest of the paper is organized as follows. In Section 2 we present a general model and introduce necessary and sufficient conditions. A simple nonexistence example is given. In Section 3 we prove equilibrium existence results, and then we discuss several typical applications of the general model in Section 4.

2. A GENERAL EQUILIBRIUM MODEL

We first introduce some notation. Let I_k denote the set of the first k positive integers. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{Z}^n$ be the set of all integer vectors of $\mathbb{R}^n$. The vectors 0^n and 1^n are the all-zero vector and the all-one vector of $\mathbb{R}^n$, respectively. We use the notation $\mathbb{R}^n_+$ for the nonnegative orthant $\{x \in \mathbb{R}^n \mid x \geq 0^n\}$ and $\mathbb{Z}^n_+$ for the set $\{x \in \mathbb{Z}^n \mid x \geq 0^n\}$. For each $i \in I_n$, the vector $e(i)$ denotes the i-th unit vector of $\mathbb{R}^n$. For a subset B of $\mathbb{R}^n$, the notation $\mathrm{Co}(B)$ denotes the convex hull of B.

We consider a competitive economy in which m agents, denoted by I_m, exchange n indivisible goods, denoted by I_n. These goods can be houses, cars, trucks, aircrafts, computers, machines, and so on. Each agent is initially endowed with a bundle of indivisible goods, denoted by $\omega^i \in \mathbb{Z}^n_+$, and some amount of money, denoted by m_i. The preference of each agent over goods is specified by a utility function $u^i : \mathbb{Z}^n \times \mathbb{R} \to \mathbb{R}$. Let W denote the social endowment of all indivisible goods, i.e., $W = \sum_{i=1}^m \omega^i$. Each agent i has a possible consumption set $Y^i = X^i \times \mathbb{R}_+$ where X^i is a bounded subset of $\mathbb{Z}^n_+$ and contains at least ω^i, $i \in I_m$, and 0^n. It is natural to require that X^i is bounded since in reality no agent would like to have an infinite amount of indivisible goods.

For each $i \in I_m$ and each price vector $p \in \mathbb{R}^n_+$, the demand correspondence $D_i(p)$ is defined by

$$\begin{aligned} D_i(p) &= \{x \mid u^i(x, p^\top \omega^i + m_i - p^\top x) \\ &= \max\{u^i(y, s) \mid y \in X^i,\, s \geq 0,\, p^\top y + s \leq p^\top \omega^i + m_i\}\}. \end{aligned}$$

Definition 2.1. A list of vectors $(p, x^1, x^2, \cdots, x^m) \in \mathbb{R}^n_+ \times D_1(p) \times D_2(p) \times \cdots \times D_m(p)$ is a competitive equilibrium if $\sum_{i=1}^m x^i = W(= \sum_{i=1}^m \omega^i)$.

Let t_m be a number strictly greater than $\sum_{i \in I_m} m_i$, i.e., the total amount of money in the market. The parameter t_m will be called a *critical value.* Let $\mathcal{P}$ denote the set of $p \in \mathbb{R}^n_+$ for which there exists a list of vectors $(x^1, \cdots, x^m) \in \mathrm{Co}(D_1(p)) \times \cdots \times \mathrm{Co}(D_m(p))$ such that

$$\begin{array}{lll} \sum_{i=1}^m x_j^i & \leq W_j & \text{for } j \in I_n \text{ with } \ p_j = 0, \\ \sum_{i=1}^m x_j^i & = W_j & \text{for } j \in I_n \text{ with } \ 0 < p_j < t_m, \\ \sum_{i=1}^m x_j^i & \geq W_j & \text{for } j \in I_n \text{ with } \ p_j \geq t_m. \end{array} \tag{1}$$

Clearly, $\mathcal{P}$ includes the set of all possible equilibrium price vectors. To ensure the existence of equilibrium, we impose the following assumptions on the economy:

A1 For each $i \in I_m$ and each $x \in \mathbb{Z}^n$, $u^i(x, \cdot)$ is a continuous and nondecreasing function.

A2 For each $i \in I_m$ and each $s \in \mathbb{R}_+$, $x \geq y$ implies $u^i(x, s) \geq u^i(y, s)$, for any $x, y \in X^i$.

A3 For each $i \in I_m$, m_i is a positive real number such that $u^i(\omega^i, m_i) \geq \max_{x \in X^i} u^i(x, 0)$.

A4 If the set $\mathcal{P}$ is nonempty, then there exists some $p \in \mathcal{P}$ such that there exists a list of vectors $(y^1, \cdots, y^m) \in D_1(p) \times \cdots \times D_m(p)$ also satisfying the inequalities (1).

Assumption A1 needs no explanation. This assumption is weaker than those made in the literature. Assumption A2 says that all goods are desirable. Free disposal is a sufficient condition for this assumption. Assumption A3 implies that the initial endowment is at least as good as any state without consumption. These three assumptions are quite natural and almost minimal. Assumption A4 says that if a system of linear inequalities (2.1) has a solution in the convex hull $\mathrm{Co}(D_1(p)) \times \cdots \times \mathrm{Co}(D_m(p))$, then it also has an integral solution in $D_1(p) \times \cdots \times D_m(p)$. We can easily see that Assumption A4 is nessesary for an equilibrium to exist. This assumption will be referred to as *integral inequality satisfiability* (IIS) condition. In the next section we will demonstrate the existence of equilibrium under Assumptions A1 through A4. Furthermore, we will show that under Assumptions A1, A2 and A3, the economy has an equilibrium if and only if Assumption A4 holds. In this sense, Assumption A4 is also a very natural and minimal condition. To get an intuitive knowledge of Assumption A4, we will immediately give an example here. There are two agents 1 and 2 in the economy. Agent 1 initially has one dollar and one indivisible good A and agent 2 has one dollar and one indivisible good B. Their utilities are given by $u_i(S, s) = V_i(S) + s$, $i = 1, 2$, where $V_1(\emptyset) = V_2(\emptyset) = 0$,

$V_1(A) = 7/6$, $V_1(B) = 1$, $V_1(AB) = 4/3$, $V_2(A) = 1/3$, $V_2(B) = 1/4$, $V_2(AB) = 5/6$. Although Assumptions A1, A2 and A3 are satisfied here, the economy has no equilibrium at all. In fact, the price vector $\bar{p} = (p_A, p_B)$ with $p_A = 1/2$ and $p_B = 1/3$ is the only possible equilibrium price vector for the economy, i.e. $\mathcal{P} = \{\bar{p}\}$. At $\bar{p}$, we have $D_1(\bar{p}) = \{\{A\}, \{B\}\}$ and $D_2(\bar{p}) = \{\emptyset, \{A, B\}\}$. Clearly, Assumption A4 is not satisfied.

3. EXISTENCE OF AN EQUILIBRIUM

In this section we will show the existence of an equilibrium for the economy as described in the last section under the four assumptions. Define $\psi(p) = \sum_{i=1}^m D_i(p) - \{W\}$ and $\phi(p) = \sum_{i=1}^m \mathrm{Co}(D_i(p)) - \{W\}$ for all $p \in \mathbb{R}^n_+$, where recall that $W = \sum_{i=1}^m \omega^i$. Let $\bar{m}$ be a number no less than the critical value t_m. Define an n-dimensional cube $C^n(\bar{m})$ by

$$C^n(\bar{m}) = \{p \in \mathbb{R}^n_+ \mid p_l \leq \bar{m} \text{ for all } l \in I_n\}.$$

Now we are ready to present our main theorem which states the existence of an equilibrium and tells its location.

THEOREM 3.1. *Under Assumptions* A1 *through* A4*, the economy has at least one competitive equilibrium with its equilibrium vector $p^* \in C^n(\bar{m})$.*

Proof. The proof will be divided into three steps:

Step 1: We will show that under Assumptions A1, A2 and A3 the correspondence ϕ is an upper semi-continuous point-to-set mapping with nonempty, convex and compact values. To prove this statement, we first show that for each $i \in I_n$, D_i is nonempty-valued upper semi-continuous on $C^n(\bar{m})$. Take an arbitrary point $\bar{p}$ from $C^n(\bar{m})$. Clearly, $D_i(\bar{p})$ is nonempty. Let $B(p, \delta)$ denote the δ-neighborhood of a point p. Take any point p^* from $C^n(\bar{m})$. Let $\{p^h\}$ be a sequence of points in $C^n(\bar{m})$ converging to p^* and $\{x^h\}$ be a sequence of points in X^i converging to x^* such that $x^h \in D_i(p^h)$ for each h. To show the upper semi-continuity of D_i, we only need to prove $x^* \in D_i(p^*)$. First notice that since X^i is bounded (and hence is a finite set), it is easy to see that there exists a positive integer M such that $x^h = x^*$ for all $h \geq M$. Secondly, note that $p^{l\top} x^l \leq p^{l\top} \omega^i + m_i$ for all l. It follows that $p^{*\top} x^l \leq p^{*\top} \omega^i + m_i$. Suppose to the contrary that $x^* \notin D_i(p^*)$, then it holds

$$u^i(x^*, p^{*\top}(\omega^i - x^*) + m_i) < u^i(x, p^{*\top}(\omega^i - x) + m_i) \tag{2}$$

for all $x \in D_i(p^*)$. It will be shown that there exists a point $\bar{x} \in D_i(p^*)$ such that $p^{*\top}\bar{x} < p^{*\top}\omega^i + m_i$. Obviously, $p^{*\top}x \leq p^{*\top}\omega^i + m_i$ for all $x \in D_i(p^*)$.

Suppose to the contrary that $p^{*\top}x = p^{*\top}\omega^i + m_i$ for all $x \in D_i(p^*)$. Then Assumption A3 implies

$$u^i(x^*, p^{*\top}(\omega^i - x^*) + m_i) < u^i(x, p^{*\top}(\omega^i - x) + m_i) = u^i(x, 0) \leq u^i(\omega^i, m_i)$$

for all $x \in D_i(p^*)$. Hence $\omega^i \in D_i(p^*)$. This, together with $m_i > 0$ from Assumption A3, means that $\omega^i \in D_i(p^*)$ and $p^{*\top}\omega^i < p^{*\top}\omega_i + m_i$. It contradicts the assumption that $p^{*\top}x = p^{*\top}\omega^i + m_i$ for all $x \in D_i(p^*)$. We thus have proved that there exists a point $\bar{x} \in D_i(p^*)$ such that $p^{*\top}\bar{x} < p^{*\top}\omega^i + m_i$. Then by Assumption A1 and Equation (2) there exists a $\delta > 0$ such that $p \in B(p^*, \delta)$ implies $p^\top \bar{x} < p^\top \omega^i + m_i$ and $u^i(x^*, p^\top(\omega^i - x^*) + m_i) < u^i(\bar{x}, p^\top(\omega^i - \bar{x}) + m_i)$. Hence there exists a positive integer $\bar{M} \geq M$ such that for all $h \geq \bar{M}$, $p^h \in B(p^*, \delta)$, $x^h = x^*$ and $u^i(x^h, p^{h\top}(\omega^i - x^h) + m_i) < u^i(\bar{x}, p^{h\top}(\omega^i - \bar{x}) + m_i)$. This implies $x^h \notin D_i(p^h)$ for all $h \geq \bar{M}$, yielding a contradiction.

Now it is known that the sum of a finite number of upper semi-continuous correspondences is upper semi-continuous. Therefore ψ is upper semi-continuous. Since for each $p \in C^n(\bar{m})$ the set $\psi(p)$ contains only a finite number of elements, $\phi(p)$ is closed. Again it is known that the convexified upper semi-continuos correspondence is still upper semi-continuous. The conclusion now follows immediately.

Step 2: We will show that $\mathcal{P}$ is nonempty. Note that ϕ satisfies the conditions of the stationary point theorem; see e.g., Hartman and Stampacchia (1966). Thus there exist a point $\bar{p} \in C^n(\bar{m})$ and a list of vectors $(y^1, y^2, \cdots, y^m) \in \mathrm{Co}(D_1(\bar{p})) \times \mathrm{Co}(D_2(\bar{p})) \times \cdots \times \mathrm{Co}(D_m(\bar{p}))$ such that

$$\begin{array}{lll} \sum_{i=1}^m y_j^i & \leq W_j & \text{if } \ \bar{p}_j = 0 \\ \sum_{i=1}^m y_j^i & = W_j & \text{if } \ 0 < \bar{p}_j < \bar{m} \\ \sum_{i=1}^m y_j^i & \geq W_j & \text{if } \ \bar{p}_j = \bar{m}. \end{array} \tag{3}$$

Recall $\bar{m} \geq t_m$. Clearly, $\mathcal{P}$ is nonempty.

Step 3: We will show that there is an equilibrium. Since $\mathcal{P}$ is a nonempty set, Assumption A4 holds for some $p^* \in \mathcal{P}$. It follows from Assumption A4 that there exists a list of vectors $(x^1, x^2, \cdots, x^m) \in D_1(p^*) \times D_2(p^*) \times \cdots \times D_m(p^*)$ satisfying the following inequalities:

$$\begin{array}{lll} \sum_{i=1}^m y_j^i & \leq W_j & \text{if } \ p_j^* = 0 \\ \sum_{i=1}^m y_j^i & = W_j & \text{if } \ 0 < p_j^* < t_m \\ \sum_{i=1}^m y_j^i & \geq W_j & \text{if } \ p_j^* \geq t_m. \end{array} \tag{4}$$

Let $I^1 = \{l \in I_n \mid \sum_{i=1}^m x_l^i < W_l\}$, $I^2 = \{l \in I_n \mid \sum_{i=1}^m x_l^i = W_l\}$ and $I^3 = \{l \in I_n \mid \sum_{i=1}^m x_l^i > W_l\}$. Notice that (I^1, I^2, I^3) is a partition of I_n. It follows from Equation (4) that $l \in I^1$ implies $p_l^* = 0$, $l \in I^2$ implies

$p_l^* \geq 0$, and $l \in I^3$ implies $p_l^* \geq t_m$. We will show that I^3 is empty. Suppose to the contrary that I^3 is nonempty. Since $(p^*)^\top x^i \leq (p^*)^\top \omega^i + m_i$ for all $i \in I_m$, we have

$$\begin{aligned}
&(p^*)^\top (\sum_{i=1}^m x^i - \sum_{i=1}^m \omega^i) \\
&= \sum_{l \in I^1} p_l^* (\sum_{i=1}^m x_l^i - W_l) + \sum_{l \in I^2} p_l^* (\sum_{i=1}^m x_l^i - W_l) + \sum_{l \in I^3} p_l^* (\sum_{i=1}^m x_l^i - W_l) \\
&= \sum_{l \in I^3} p_l^* (\sum_{i=1}^m x_l^i - W_l) \leq \sum_{i=1}^m m_i.
\end{aligned}$$

Hence we have $\sum_{l \in I^3} p_l^* \leq \sum_{l \in I^3} p_l^* (\sum_{i=1}^m x_l^i - W_l) \leq \sum_{i=1}^m m_i$. It follows that $p_l^* \leq \sum_{i=1}^m m_i$ for all $l \in I^3$. However, $p_l^* \geq t_m > \sum_{i=1}^m m_i$ for all $l \in I^3$, which yields a contradiction.

So far we have proved that $\sum_{i \in I_m} x_j^i = W_j$ whenever $p_j^* > 0$. We still have to show that there is a list of vectors $(x^{*1}, \cdots, x^{*m}) \in D_1(p^*) \times \cdots \times D_m(p^*)$ such that $\sum_{i=1}^m x^{*i} = W$. Suppose that the set I^1 is nonempty. Let (y^i)'s be nonnegative integer vectors such that $\sum_{i=1}^n y^i = W - \sum_{i=1}^n x^i$ and $x^i + y^i \in X^i$. Let $x^{*i} = x^i + y^i$ for each $i \in I_m$. Since $p_j^* = 0$ for $j \in I^1$ and $y_j^i = 0$ for $j \in I_n \setminus I^1$, we have $(p^*)^\top x^{*i} = (p^*)^\top x^i$ for all $i \in I_m$. It follows from Assumption A2 that $u^i(x^{*i}, (p^*)^\top(\omega^i - x^{*i}) + m_i) = u^i(x^{*i}, (p^*)^\top(\omega^i - x^i) + m_i) \geq u^i(x^i, (p^*)^\top(\omega^i - x^i) + m_i)$. This implies that $x^{*i} \in D_i(p^*)$ for all $i \in I_m$. By the definition of (y^i)'s it is clear that $\sum_{i=1}^m x^{*i} = W$. We are done. ▌

Following the proof of Theorem 3.1, we immediately have the following two results.

THEOREM 3.2. *Under Assumptions* A1 *through* A3*, the economy has a competitive equilibrium with its equilibrium vector* $p^* \in \mathbb{R}_+^n$ *if and only if Assumption* A4 *holds.* ▌

If utility function u^i is quasi-linear in money (i.e., $u^i(x, s) = f^i(x) + s$ with $f^i(0^n) = 0$), we will call f^i a *reservation value function.* In the quasi-linear case, we have

THEOREM 3.3. *Suppose that each agent* $i \in I_m$ *has a quasi-linear utility in money* $u^i(x, s) = f^i(x) + s$ *with* $f^i(0^n) = 0$*, and* f^i *is weakly increasing (i.e., for any* $x, y \in X^i$*,* $x \leq y$ *implies* $f^i(x) \leq f^i(y)$*), and that* m_i *is a positive real number such that* $m_i \geq \max_{x \in X^i} f^i(x) - f^i(\omega^i)$*. Then the economy has a competitive equilibrium with its equilibrium vector* $p^* \in \mathbb{R}_+^n$ *if and only if Assumption* A4 *holds.* ▌

Theorem 3.3 has somehow generalized the earlier results of Bikhchandani and Mamer (1997) since the model here is slightly more general than theirs. We can summarize their model here. There are one seller and m buyers in the market. The seller (in fact a fictitious agent) has a set of indivisible objects, denoted by $\{e(1), \cdots, e(n)\}$, with some money, and his utility function is $u^0(x, m) = m$, while the m buyers have no initial endowment of objects but money with $m_i > f^i(1^n)$, $i \in I_m$. Each buyer has a quasi-linear utility in money $u^i(x, m) = f^i(x) + m$ with $f^i(0^n) = 0$, where reservation function f^i is weakly increasing. The consumption set X^i of each agent is given by $X^i = \{x \in \mathbb{Z}^n_+ \mid x \leq 1^n\}$. They introduced a necessary and sufficient condition for the existence of an equilibrium. For a similar model, Bevia et al. (1999) proved the existence of an equilibrium if reservation functions f^i are weakly increasing submodular and have cardinality property. They proved such reservation functions satisfy the well known gross substitute condition proposed by Kelso and Crawford (1982). We will come back to this point in the subsequent section.

4. APPLICATIONS

In this section we will show several applications of the model presented in Section 2. Before doing this, we first review several concepts from the field of discrete optimization. Given a set I_n, we call a set function $f : 2^{I_n} \to \mathbb{R}$ a *submodular function* on 2^{I_n} if it satisfies

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T), \quad \forall S, T \subseteq I_n. \tag{5}$$

We assume from now on that $f(\emptyset) = 0$ for any set function f; see Fujishige (1991) for more detail. The *generalized polymatroid* (or g-polymatroid), introduced by Frank (1984), is defined by

$$\mathrm{P}(f, g) = \{x \in \mathbb{R}^n \mid \forall S \subseteq I_n : g(S) \leq x(S) \leq f(S)\}, \tag{6}$$

where f and $-g$ are submodular functions satisfying the following condition:

$$f(S) - g(T) \geq f(S \setminus T) - g(T \setminus S), \quad \forall S, T \subseteq I_n.$$

A nonempty polyhedron is said to be *integral* if each of its nonempty faces contains an integral point. It is known that a g-polymatroid $\mathrm{P}(f, g)$ is integral if f and g have only integer values, and that if P and Q are integral g-polymatroids in $\mathbb{R}^n$, then $\mathrm{P} + \mathrm{Q}$ is also an integral g-polymatroid and it holds that

$$(\mathrm{P} + \mathrm{Q}) \cap \mathbb{Z}^n = (\mathrm{P} \cap \mathbb{Z}^n) + (\mathrm{Q} \cap \mathbb{Z}^n).$$

See Danilov et al. (2001) and Murota and Tamura (2001) for a related application of polymatroids.

Introduced by Murota and Shioura (1999) and Murota (1998), a function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with the effective domain $\mathrm{dom}(g) = \{z \in \mathbb{Z}^n \mid g(z) > -\infty\}$ being nonempty and bounded is called *M$^\natural$-concave* if it satisfies the following condition:

($M^\natural$) For $x, y \in \mathrm{dom}(g)$ and $k \in \mathrm{supp}^+(x-y)$,

$$g(x) + g(y) \leq$$
$$\max[g(x-e(k)) + g(y+e(k)), \max_{l \in \mathrm{supp}^-(x-y)} \{g(x-e(k)+e(l)) + g(y+e(k)-e(l))\}],$$

where $\mathrm{supp}^+(x-y) = \{k \in I_n \mid x_k > y_k\}$ and $\mathrm{supp}^-(x-y) = \{k \in I_n \mid x_k < y_k\}$.

An M$^\natural$-concave function is related to g-polymatroids as shown in Murota and Shioura (1999).

THEOREM 4.1. *A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with the effective domain* $\mathrm{dom}(g) = \{z \in \mathbb{Z}^n \mid g(z) > -\infty\}$ *being nonempty and bounded is M$^\natural$-concave if and only if g satisfies the following conditions:*
(i) *the convex hull of the effective domain* $\mathrm{dom}(g)$ *is an integral g-polymatroid;*
(ii) *For each $p \in \mathbb{R}^n$, defining*

$$Y(p) = \arg\max\{g(z) - p^\top z \mid z \in \mathbb{Z}^n\} \tag{7}$$

then the convex hull $\mathrm{Co}(Y(p))$ *of $Y(p)$ in $\mathbb{R}^n$ is an integral g-polymatroid and it holds* $Y(p) = \mathrm{Co}(Y(p)) \cap \mathbb{Z}^n$. ▮

With respect to the concept of integral g-polymatroid, we have the following simple but quite useful lemma.

LEMMA 4.1. *If for each $i \in I_m$ and $p \in \mathbb{R}^n_+$,* $\mathrm{Co}(D_i(p))$ *is an integral g-polymatroid and* $D_i(p) = \mathrm{Co}(D_i(p)) \cap \mathbb{Z}^n$, *then Assumption* A4 *is satisfied.*

Proof. Suppose that there exist a partition (J_1, J_2, J_3) of V and a list of vectors $(x^1, \cdots, x^m) \in \mathrm{Co}(D_1(p)) \times \cdots \times \mathrm{Co}(D_m(p))$ such that

$$\begin{array}{lll} \sum_{h=1}^m x_j^h \leq W_j & \text{for} & j \in J_1 \\ \sum_{h=1}^m x_j^h = W_j & \text{for} & j \in J_2 \\ \sum_{h=1}^m x_j^h \geq W_j & \text{for} & j \in J_3 \end{array} \tag{8}$$

Since for each $i \in I_m$ $\mathrm{Co}(D_i(p))$ is an integral g-polymatroid, the sum $\sum_{i \in I_m} \mathrm{Co}(D_i(p))$ is also an integral g-polymatroid and it holds that

$$\{\sum_{i \in I_m} \mathrm{Co}(D_i(p))\} \cap \mathbb{Z}^V = \sum_{i \in I_m} D_i(p).$$

Let B be a box in $\mathbb{R}^V$ defined by the direct product of intervals

$$\begin{array}{ll} [0, W_j] & \text{for } j \in J_1 \\ [W_j, W_j] & \text{for } j \in J_2 \\ [W_j, +\infty) & \text{for } j \in J_3 \end{array} \tag{9}$$

From (8) $\{\sum_{i \in I_m} \mathrm{Co}(D_i(p))\} \cap B$ is nonempty and it is also an integral g-polymatroid since a nonempty intersection of an integral g-polymatroid and an integral box is an integral g-polymatroid. It follows that there exists a list of integral vectors $(y^1, \cdots, y^m) \in D_1(p) \times \cdots \times D_m(p)$ satisfying the inequalities (8). ▌

We now consider the models presented by Quinzii (1984), Gale (1984), Svensson (1984), Kaneko and Yamamoto (1986) in which there are the same number of indivisible objects as agents. It is easy to check that Assumptions A1, A2 and A3 are satisfied by their conditions. It remains to see how Assumption A4 is also satisfied by their conditions. Notice that in their models all agents have the same consumption set $X^i = \{0^n, e(1), e(2), \cdots, e(n)\}$, $i \in I_n$. Since the convex hull of any subset of X^i is an integral g-polymatroid, the convex hull of the demand set of agent i, $\mathrm{Co}(D_i(p))$, must be also an integral g-polymatroid. It follows from Lemma 4.1 that Assumption A4 is satisfied.

When each agent has a quasi-linear utility function $u^i(x, s) = f^i(x) + s$ with $f^i(0^n) = 0$ and f^i is M$^\natural$-concave. We have

THEOREM 4.2. *Suppose that each agent $i \in I_m$ has an initial endowment (ω^i, m_i) and has a quasi-linear utility in money $u^i(x, s) = f^i(x) + s$ with $f^i(0^n) = 0$, and f^i is a weakly increasing M$^\natural$-concave function on the consumption set $X = \{x \in \mathbb{Z}^n \mid 0^n \le x \le \sum_{i \in I_m} \omega^i\}$ and that m_i is a positive real number such that $m_i \ge f^i(W) - f^i(\omega^i)$. Then the economy has a competitive equilibrium with its equilibrium vector $p^* \in \mathbb{R}^n_+$.*

Proof. Clearly, Assumptions A1, A2 and A3 are satisfied. We need to show that Assumption A4 is also satisfied. To do so, we will show that the convex hull of the demand set $D_i(p)$ of each agent i is an integral g-polymatroid and $D_i(p) = \mathrm{Co}(D_i(p)) \cap \mathbb{Z}^n$. Note that

$$D_i(p) = \{x \mid f^i(x) - p^\top x = \max\{f^i(y) - p^\top y \mid y \in X,\, p^\top y \le p^\top \omega^i + m_i\}\}.$$

Because of the budget constraint, $\mathrm{Co}(D_i(p))$ may not be an integral g-polymatroid. Consider the following set

$$Y_i(p) = \{x \mid f^i(x) - p^\top x = \max_{y \in X}(f^i(y) - p^\top y)\}.$$

Since f^i is M$^\natural$-concave, by Theorem 4.1 Co$(Y_i(p))$ is an integral g-polymatroid and $Y_i(p) = \text{Co}(Y_i(p)) \cap \mathbb{Z}^n$. We will show $Y_i(p) = D_i(p)$. Suppose that $x^* \in Y_i(p)$. Then we have $f^i(x^*) - p^\top x^* \geq f^i(\omega^i) - p^\top \omega^i$. It follows from Assumptions A2 and A3 that

$$m_i + f^i(\omega^i) - p^\top x^* \geq f^i(W) - p^\top x^* \geq f^i(\omega^i) - p^\top \omega^i.$$

Now it is easy to see that $p^\top x^* \leq p^\top \omega^i + m_i$. Hence $x^* \in D_i(p)$. We are done. ▮

Finally we consider a modified version of the model presented by Kelso and Crawford (1982). In this model each agent initially owns a set N_i of indivisible objects with certain amount of money m_i. Of course, some N_i might be an empty set. Let I_n denote the set of all indivisible objects in the economy. Each agent is assumed to have a quasi-linear utility function $u_i(S, s) = f^i(S) + s$ with $f^i(\emptyset) = 0$. The reservation value function f^i is weakly increasing, i.e., for any $S, T \subseteq I_n$, $f^i(S) \leq f^i(T)$. Then the demand set $D_i(p)$ can be written as

$$\begin{aligned} D_i(p) &= \{S \mid f^i(S) - \sum_{h \in S} p_h \\ &= \max\{f^i(T) - \sum_{h \in T} p_h \mid T \subseteq I_n, \sum_{h \in T} p_h \leq \sum_{h \in N_i} p_h + m_i\}\}. \end{aligned}$$

From the definition of $D_i(p)$, we see that the consumption set for each agent is the set $X = \{x \in \mathbb{Z}^n \mid \mathbf{0} \leq x \leq \sum_{i \in I_n} e(i)\}$. We will show that there exists an equilibrium if Assumptions A2 and A3 and the following condition hold.

(GS) For any two price vectors p and q such that $p \leq q$, and any $A \in D_i(p)$, there exists $B \in D_i(q)$ such that $\{i \in A \mid p_i = q_i\} \subseteq B$.

Assumption (GS) says that given a price p, if agent i chooses object j in his choice set, then agent i will still want to choose object j in his new choice set when the prices of other objects rise, but the price of object j remains the same. This assumption can be seen as an extended version of the well-known *gross substitutes* (GS) condition proposed by Kelso and Crawford (1982) in which they impose no budget constraints. As pointed out in Roth and Sotomayor (1990, p.183), the introduction of budget constraints may cause the core to be empty. Nevertheless, as shown above in Theorem 4.2, the budget constraints are redundant in determining the demand correspondences if m_i is a positive real number such that $m_i \geq f^i(I_n) - f^i(N_i)$.

THEOREM 4.3. *Suppose that each agent $i \in I_m$ has a quasi-linear utility in money $u^i(S, s) = f^i(S) + s$ with $f^i(\emptyset) = 0$, and f^i is a weakly* *increasing function and that m_i is a positive real number such that $m_i \geq f^i(I_n) - f^i(N_i)$. Then the economy has a competitive equilibrium with its equilibrium vector $p^* \in \mathbb{R}^n_+$, if f^i satisfies the gross substitutes condition* (GS) *for every $i \in I_m$.* ▮

The following theorem has been shown by Fujishige and Yang (2000).

THEOREM 4.4. *A weakly increasing reservation value function f^i satisfies the gross substitutes condition* (GS) *if and only if f^i is $M^\natural$-concave.*

Then it is easy to see the conditions stated in Theorem 4.3 satisfy those stated in Theorem 4.2 and thus there exists an equilibrium. Note that in the above theorem the effective domain of each reservation value function f^i is given by 2^{I_n}.

Finally we mention that two special classes of reservation value functions with the GS property are found by Kelso and Crawford (1982) and Bevia et al.(1999). Furthermore, Gul and Stacchetti (1999) introduce two new conditions which are equivalent to the GS condition.

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