ANNALS OF ECONOMICS AND FINANCE **2**, 379–400 (2001)

Bankruptcy Law, Capital Allocation, and Aggregate Effects: A Dynamic Heterogenous Agent Model with Incomplete Markets

Tao Zha*

Research Department
Federal Reserve Bank of Atlanta
Atlanta , GA 30330-2713, USA

Under the assumption that asset markets are incomplete, this paper introduces bankruptcy in an intertemporal heterogenous agent model with capital accumulation and heterogeneous agents. It explores the role of regulatory intervention and argues that intervention in the form of a level of bankruptcy exemption can enhance not only social welfare but also distributive equity. The bankruptcy law is carefully specified in the model. The model generates distributional changes in consumption, capital, and bankruptcy risk in response to an adjustment in the exemption level and accentuates the effects of these redistributions on aggregate variables.

Key Words: Bankruptcy law; Capital allocations; Heterogenous agent; Incomplete markets.

JEL Classification Numbers: E69, D52, D92, G18

1. INTRODUCTION

In their static security models, Dubey, Geanakoplos and Shubik [1995] and Zame [1993] argue that default, as well as the probability of default, plays an important role in improving economic efficiency. Two important features account for their general equilibrium results. One is that they take market incompleteness as given, assuming that certain contingencies

*The views expressed here are those of the author and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. This paper is based in part on the author's Ph.D. dissertation, "Capital Allocation, Bankruptcy and Endogenous Growth in an Economy with Incomplete Asset Markets." He is grateful to Christopher Sims for advice and to numerous people, including William Brainard, Roberto Chang, John Geanakoplos, Ariel Pakes, William Roberds, and especially Eric Leeper, Robert E. Lucas, and Neil Wallace for helpful discussions. Any remaining errors are the author's responsibility.

1529-7373/2001

cannot be written into contracts. The second is that they impose in their models the exogenously determined default penalties. In this paper we use versions of these features to introduce bankruptcy in a dynamic stochastic model with capital accumulation in order to shed some light upon the role of regulatory intervention in improving social welfare and in redistributing individual wealth.

Our general equilibrium model is built upon the standard growth model (e.g., Brock and Mirman [1972], Blanchard and Fischer [1989]) modified to account for a role of bankruptcy in an environment of incomplete markets with heterogeneous agents. Specifically, we consider an economy composed of a continuum of entrepreneurs who are subject to idiosyncratic production shocks. We assume that entrepreneurs' own idiosyncratic shocks can be verified at a cost. We discuss how bankruptcy law is specified in this environment. Assuming that asset markets are incomplete, we postulate a standard loan contract and work out its various implications on the steady state equilibrium behavior of our model economy. We impose an exemption level as the limit up to which a borrower can discharge the debts and exempt the assets in the state of bankruptcy. Similar to Dubey, Geanakoplos and Shubik [1995], we view the exemption level as exogenously determined by bankruptcy law. The law specified in this paper permits a borrower to write a new debt contract with the intermediary after the event of bankruptcy. Unlike the optimal contracting theory, our approach, in spirit of other incomplete markets models (e.g., Dubey, Geanakoplos and Shubik [1995], Zame [1993]), is analogous to the cash-in-advance model in which the payment mechanism is taken as and institutional arrangement – another form of incomplete markets.[1] The approach of incomplete markets in our model gives one the freedom to explore the role of regulatory intervention such as bankruptcy law in improving efficiency and distributive equity.

Our results manifest the five key features of this exercise: (1) incomplete markets, (2) bankruptcy, (3) capital accumulation, (4) distributional changes, and (5) aggregate effects. We emphasize how an adjustment in the exemption level influences capital reallocation and changes the distributions of consumption and bankruptcy risk in a dynamic general equilibrium model. With the aggregate constraint that the zero-profit intermediary must balance its budget every period, we obtain a stationary distribution of wealth across heterogenous agents in the steady state equilibrium and derive the risk-free interest rate endogenously. Our quantitative results[2] suggest that an adjustment in the exemption level changes wealth distri-

[1]The idea of imposing a cash-in-advance constraint can be traced to Clower [1967].

[2]We apply the minimum weighted residual approach discussed in Judd [1992] to our model computation. Although this computation method is non-standard, it proves efficient for our problem, especially when our model features enable us to reduce a set of state variables to only one dimension.

bution and that such adjustment can be welfare improving. Moreover, we explore the implications of distributive equity by examining the length of time it takes for the "poor" to become "rich" as well as for the rich to become poor.

While the contribution of this paper is theoretical, the model is motivated largely by recent experience with actual bankruptcy laws. For some countries and in certain time periods governments pass laws regulating the form of contracts (Aghion, Hart and Moore [1992]). A useful example is the U.S. Bankruptcy Code of 1978 that establishes generous bankruptcy exemption standards allowing debtors to discharge part of the debts and exempt some of the assets (Shepard [1984], Boyes and Faith [1986]). Although a variety of possible explanations for this kind of law are beyond the scope of this paper, we note that there may be a social interest in enforcing a level of bankruptcy exemption, assuming that public assistance funds relieves the pain and suffering of innocent paupers (victims of unfortunate events) (as to detailed arguments for such an assumption, see Baumol [1986], Zajac [1986]). Legal enforcement of more detailed and contingent contracts may be impractical or prohibitively expensive (Dubey, Geanakoplos and Shubik [1995], Calomiris and Hubbard [1990], Zame [1993]). Our model is designed to reflect certain features we observe in the U.S. economy: the rates individuals pay on loans vary with their wealth; an individual's assets can be exempted up to the exemption level in the state of bankruptcy; the exemption level is regulated by bankruptcy law; an *ex-post* verification in the state of bankruptcy is straightforward in many bankruptcy cases (Calomiris and Hubbard [1990]).

Although the intertemporal model we construct to formulate these ideas is somewhat complicated, one of the major results can be understood in a simple one-period model presented in Section 2. This result is that intervention in the form of an exemption level can be welfare enhancing and that there is a nonmonotone relationship between welfare and the exemption level. The intuition we gain from this simple model is helpful to understanding the main model discussed in the rest of this paper. Section 3 offers an exposition of our dynamic heterogeneous agent model with capital accumulation. In Section 4, we obtain a number of tentative results for the stochastic steady state equilibrium and discuss their various implications.

2. A SIMPLE CASE

In this section, we consider a simple two-person, one-date model to help us gain intuition on a nonmonotone relation of utility to an exemption level.[3] The two person are classified as a "borrower" and a "lender". The

[3]I am indebted to Neil Wallace for providing this model.

lender is risk-neutral, and has an endowment (denoted by e) and no investment project; the borrower is risk-averse, and has a risky project and no endowment. The utility function of the borrower is logarithmic. With the amount of an input l, the project yields a random return ηl^{α} where $0 < \alpha < 1$ and η takes two values – g with probability π_g and b with π_b. The return on a project is freely observed only by the borrower, but the lender is able to verify the state at the cost proportional to the amount of lending l. We denote this cost by $\varkappa l$. Further, we denote the gross loan rate by r. Finally, we assume that the lender has free access to a risk-free return ρ on its endowment.

A loan contract between the borrower and the lender consists of both the loan volume l and the loan rate r, and is subject to the law's imposition on an exemption level (denoted by $\overline{w}$) that applies to the state in which the borrower is unable to repay the debt rl in full. This state will be verified and the lender will get everything above the exemption level. Specifically, if $\eta l^{\alpha} - rl < \overline{w}$ and $\eta l^{\alpha} > \overline{w}$, the lender collects the residual $(\eta l^{\alpha} - \overline{w})$ less the verification cost $\varkappa l$; if $\eta l^{\alpha} \leq \overline{w}$, the lender gets nothing at the cost of $\varkappa l$. For the lender, the expected return on lending must be no less than the risk-free return. This contractual constraint can be written as

$$\rho l \leq P(\eta \geq x) rl + (\pi_g g \iota(y < g < x) + \pi_b b \iota(y < b < x)) l^{\alpha}$$

$$- P(y < \eta < x)\overline{w} - P(\eta < x)\varkappa l \tag{1}$$

where $x = (rl + \overline{w})/l^{\alpha}$, $y = \overline{w}/l^{\alpha}$, $P()$ is the probability of the event in parentheses, and $\iota()$ is an indicator function returning 1 when the statement in parentheses is true and 0 otherwise. Since the lender verifies the state of bankruptcy, the contract specified this way is incentive-compatible in the sense that the borrower has no incentive to declare a false state.

The form of contracts we specify here is similar to that in Gale and Hellwig [1985]. But the difference is that we emphasize how the equilibrium changes as a function of intervention variable $\overline{w}$, while Gale and Hellwig [1985], as well as other papers concerning bankruptcy (e.g., Smith [1972], Hellwig [1977], de Meza and Webb [1987], Calomiris and Hubbard [1990], Moore [1993]), assume that the person who defaults gets nothing – a situation analogous to our case of $\overline{w}$ being zero.

From a menu of the contracts we have described, the borrower chooses a pair (r, l) so as to maximize the expected utility function. If we denote $\eta l^{a} - rl$ by c^{*}, the maximization problem can be summarized as

$$\max_{(r,l)} E \log(c)$$

TABLE 1.

Results or the Simple Case

$\overline{w}$	U	l	r
0.00	-1.16	0.10	1.00
0.20	-1.03	0.18	1.46
0.50	-0.91	0.22	2.01
0.53	-1.07	0.07	3.52

subject to (1), $0 < l \leq e$, and

$$c = \iota(c^* > \overline{w})c^* + \iota(c^* \leq \overline{w} \text{ and } \eta l^\alpha \geq \overline{w})\overline{w} + \iota(\eta l^\alpha < \overline{w})\eta l^\alpha.$$

Constraint (1) is always binding because a lower loan rate makes the borrower strictly better off, other things being the same. To obtain the interior solution to the problem, we let the value of e be large enough so as to leave the constraint ($l \leq e$) unbinding. This optimization problem is standard, and we calibrate it with the following parameter values: $\alpha = 0.3$, $\pi_g = \pi_b = 0.5$, $\rho = 1$, $\chi = 0.01$, $g = 1.5$, and $b = 0.5$. The relationship between the expected utility and the exemption level is shown in Table 1.

As indicated in Table 1, all the cases bar the zero exemption involve a risky debt. We see that an increase in moderate exemptions (from 0 to 0.2 and from 0.2 to 0.5), while permitting a debt to be riskier, encourages borrowing and raises the expected utility. This finding is intuitive because the borrower is risk-averse and a reasonable exemption imposed by law serves as an insurance against disastrous events. Too large an exemption (e.g., when $\overline{w} = 0.53$), however, forces the lender to restrict lending in order to match the expected loan return to the risk-free rate, and thus becomes welfare reducing.

The equilibrium results for (r, l) can be also understood by considering how a change of $\overline{w}$ shifts supply and demand in the loan market.[4] An increase in $\overline{w}$ reduces the supply of loans while increasing the demand for loans. The net effect is that the loan rate r is an increasing function of $\overline{w}$ as we see in Table 1. But the equilibrium loan volume l is a nonmonotone function of $\overline{w}$. An increase in moderate exemptions shifts the demand function more than the supply function in the sense that the resulting equilibrium loan volume increases. On the other hand, too large an exemption shifts the supply function more than the demand function so that the resulting loan volume declines.

[4]One could think of equation (1) as "supply function" (which is upward sloping) and the derived Euler equation from the maximizaiton problem as "demand fuction" (which is downward sloping).

The intuition we gain in the previous paragraphs has two benefits. First, it suggest a relationship between welfare and the exemption level which is developed more fully in our intertemporal model. Second, it helps us understand some of the difficulties we will encounter from an intetemporal model with heterogeneous agents. The difficulties arise mainly from the fact that the distributions of individual variables shift in response to a change of $\overline{w}$ and those redistributions have material effects on aggregate variables. In particular, an increase in moderate exemptions may reallocate accumulated capital stock in such a way that aggregate capital actually declines, and we do not have a prior belief that social welfare will necessarily improve when the aggregate capital falls.

3. THE DYNAMIC HETEROGENEOUS AGENT MODEL WITH BANKRUPTCY LAW

A. Environment

Our general equilibrium model is built on the following environment.

Agents. The economy is composed of a continuum of infinitely lived agents called "entrepreneurs". Each entrepreneur is both a consumer and a producer.

Goods and Assets. There is only one kind of goods in this economy, which can be either consumed or invested in various assets. these assets include the physical capital stock "k" that is used to produce goods, and the bond "b" when an intermediation takes place.

Preference. Each entrepreneur is risk averse and has the same preference represented by

$$E_0 \sum_{t=1}^{\infty} \beta^t u(c_t) \tag{2}$$

where c is the entrepreneur's consumption, and $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$. Although risk aversion complicates the model, it is crucial for us to obtain a relationship between welfare and the exemption level as we have already seen from the simple model in Section 2.

Technology and Idiosyncratic Shocks. Each entrepreneur is endowed with an initial positive capital stock k_0 and with a production the technology that requires the entrepreneur's unique skill. Thus the technology of one entrepreneur is not interchangeable with that of others. The capital stock k_t depreciates exponentially at the rate of $1 - \delta$. The functional form of production, $\eta_t(Ak_{t-1}^{\alpha} + \delta k_{t-1})$, is the same for all entrepreneurs, where η_t's are i.i.d. continuous idiosyncratic shocks. We denote the density function of η_t by $f(\eta_t)$, and the corresponding distribution function by $F(\eta_t)$. There is no aggregate uncertainty.

*Autarkic Situation.*We assume that an entrepreneur can freely observe its own shock η_t, consumption c_t, and capital stock k_t at time t. We also assume that there is no technology that enables entrepreneurs to communicate with each other. In this autarkic economy, all entrepreneurs must finance their own consumption and investment. Thus the problem faced by each individual is to maximize the utility (2) subject to the budget constraint $c_t + k_t = w_t$ where $w_t = \eta_t(Ak_{t-1}^{\alpha} + \delta k_{t-1})$. The variable w_t, thought of as the entrepreneur's final wealth, is the only state variable in this dynamic problem. It distinguishes one entrepreneur (or the group of entrepreneurs measured in density) from others. Each individual dynamic problem, therefore, has a standard recursive solution.

Intermediation. Now let us consider an economy allowing intermediation with the following features. The entrepreneur's own consumption c_t, as in the autarkic case, is private information and cannot be observed by others. The capital stock k_t, however, is publicly observable in both periods t and $t+1$ (no records of k_t in other periods are required for out model). The zero-profit intermediary issues one type of bonds b_t to all lenders, but makes different types of loans l_t to different borrowers. At time t, the gross rate of return on bonds, ρ_t, is public information; it is also risk free because our model has no aggregate uncertainty. When an entrepreneur and the intermediary decide on b_t and l_t, they need to observe (or record) those data only in periods t and $t+1$. The intermediary provides the loan services at the cost of ςl_t; and it can verify the entrepreneur's own idiosyncratic shock η_t at the cost of $\varkappa l_t$. All these costs, though paid by the intermediary, are actually borne by the borrowers because the costs affect variables such as loan rates and bankruptcy risks.[5]

B. Specifications of Bankruptcy Law

This part of the section discusses how bankruptcy law is specified in our environment.

To begin with, the bankruptcy law we consider in the model applies to only two states for the entrepreneur. One is called "the state of solvency" and the other "the state of bankruptcy". These two states are mutually exclusive. The bankruptcy law requires that the state of bankruptcy be verified by the intermediary.

The contract between an entrepreneur and the intermediary at time t involves the following decisions: (i) the amount of the entrepreneur's bor-

[5]The incomplete information here concerns the observation of *ex-post* returns on an entrepreneur's investment project. There is no adverse selection or moral hazard in our model. The model could be complicated by these features, but to make the model tractable the trade-off would be to assume, like typical adverse selection model, that the projects are of fixed size (see Gale and Hellwig [1985] for more discussions of the trade-off). We note that the endogenous investment decision in the model here plays a crucial role in deriving the distribution of wealth.

rowing "l_t" and the corresponding loan rate "r_t"; (ii) a stock of the entrepreneur's accumulated capital "k_t"; (iii) a stock of the entrepreneur's bonds "b_t"; (iv) the intermediary's next-period return on its loans, denoted by R_{t+1}.[6] Since the law prescribes the intermediary's return, we shall specify the exact functional form for R_t after we expound the precise meanings of the *state of bankruptcy* and the *bankruptcy law*.

We first define the state of bankruptcy. Let us denote the entrepreneur's total assets at time t by n_t where

$$n_t = \eta_t(Ak_{t-1}^{\alpha} + \delta k_{t-1}) + \rho_{t-1}b_{t-1}. \tag{3}$$

We call the difference between total assets and debt repayments "final wealth" which is denoted by w_t whereby $w_t = n_t - R_t$.[7]

DEFINITION 3.1. For some constant $\overline{w}$, the *state of bankruptcy* is the state in which $w_t < \overline{w}$, and the state of solvency is the state in which $w_t \geq \overline{w}$.

Let us now complete the specifications of the bankruptcy law in our model. If the entrepreneur reports the state of bankruptcy, part of the debts will be discharged, and some of the total assets will be exempted up to the amount $\overline{w}$. The bankruptcy law determines the level of $\overline{w}$ exogenously, and accordingly we call $\overline{w}$ the exemption level in this paper. The debt contract is subject to this bankruptcy law, and the intermediary's return has the following features: if $n_t - r_{t-1}l_{t-1} \geq \overline{w}$, then $R_t = r_{t-1}l_{t-1}$; otherwise, $R_t = \iota(n_t \geq \overline{w})\ (n_t - \overline{w})$ where $\iota()$ is an indicator function that is defined in Section 2.[8] Such contract is incentive compatible in the sense that borrowers have no gain in reporting a false state.

With the bankruptcy law thus specified, a borrower can terminate an old contract with the intermediary and start a new one every period. The borrower may roll over the old debt through new borrowing with a newly scheduled loan rate, without being bankrupt. But since the law exempts

[6]As in Gale and Hellwig [1985], the contract written with the central zero-profit intermediary (or the mutual fund) is equivalent to the one written in competitive credit markets wherein a large number of intermediaries exist and each entrepreneur deals exclusively with one intermediary at a time.

[7]The definition of w_t is different from that of w_t in the autarkic case. As we shall see, the autarkic case is simply a special situation of our model economy with bankruptcy law.

[8]In the actual economy such as the U.S., there are many institutional considerations for these types of debt contracts. An exhaustive analysis is beyond the scope of this paper, but we note that legal enforcement of more detailed and contingent contracts (e.g., allowing the exemption level to depend on wealth,or interest rates on state) presents practical problems in court (for more discussions, see Dubey, Geanakoplos and Shubik [1989], Calomiris and Hubbard [1990]).

some of the total assets in the state of bankruptcy, declaring bankruptcy becomes the optimal strategy. Moreover, such a succession of short-term contracts requires no records of the distant past variables, and enables us to obtain the steady state equilibrium through a one-dimensional recursive problem as we will show next.[9]

C. General Characterizations of the Model

We begin by rewriting an entrepreneur's final wealth as:

$$w_t = \psi(x_t)w_t^* + (1-\psi(x_t))[\psi(y_t)\overline{w} + (1-\psi(y_t))n_t] \tag{4}$$

where

$$\begin{aligned}
&w_t^* = -r_{t-1}l_{t-1} + \eta_t(Ak_{t-1}^{\alpha} + \delta k_{t-1}) + \rho_{t-1}b_{t-1},\\
&x_t = [r_{t-l}l_{t-1} + \overline{w} - \rho_{t-1}b_{t-1}]/(Ak_{t-1}^{\alpha} + \delta k_{t-1}),\\
&y_t = [\overline{w} - \rho_{t-1}b_{t-1}]/(Ak_{t-1}^{\alpha} + \delta k_{t-1}),\\
&\psi(x_t) = \iota(w_t^* \geq \overline{w}) = \iota(\eta_t \geq x_t),\\
&\psi(y_t) = \iota(n_t \geq \overline{w}) = \iota(\eta_t \geq y_t).
\end{aligned}$$

For the zero-profit intermediary (the mutual fund), the expected return on individual loan must match the risk-free bond rate. The contractual constraint, therefore, can be written as:

$$\rho_t(1+\varsigma)l_t - \int_{y_{t+1}}^{x_{t+1}} [n_{t+1} - \overline{w}]f(\eta_{t+1})d\eta_{t+1} \tag{1}$$

$$+ \int_{0}^{x_{t+1}} \varkappa l_t f(\eta_{t+1})d\eta_{t+1} - [1 - F(x_{t+1)}]r_t l_t = 0. \tag{5}$$

For notational simplicity, we denote the left hand side term of (5) by j_t which is a function of $(l_t, r_t, k_t, b_t, \rho_t)$.

The general equilibrium model is composed of both an individual problem and the aggregate constraint. *Individually*, each entrepreneur (indexed by w_t) chooses a vector of variables $(c_t, k_t, b_t, r_t, l_t, w_t)$ to maximize utility (2) subject to constraints (4) and (5), the budget constraint

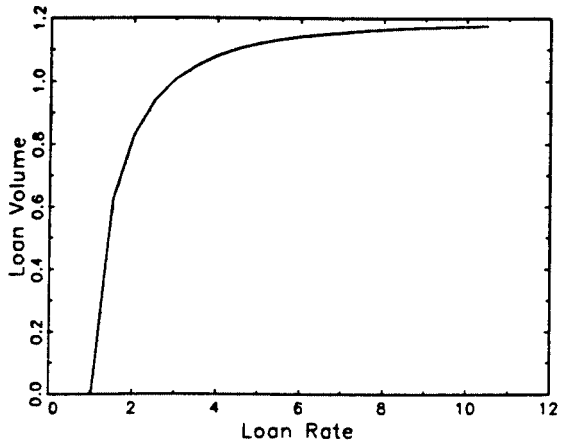
$$c_t + k_t + b_t = w_t + l_t, \tag{6}$$

and

$$b_t \geq 0, l_t \geq 0.$$

[9]Hart and Moore [1994] have recently studied a number of comparative statics properties concerning the maturity structure of the debt repayment path. Their model is nonstochastic and of a finite multi-period. Incorporation a maturity structure of long-term debt contracts in an intertemporal stochastic model will be a challenging topic for future research.

FIG. 1. Intermediary's Zero Profit Curve



In aggregation, the intermediary's budget must be balanced every period, that is to say,

$$B_t - \rho_{t-1}B_{t-1} + \rho_{t-1}(1+\varsigma)L_{t-1} - (1+\varsigma)L_t = 0, \tag{7}$$

where the capital letter B denotes an aggregation of bonds and L aggregation of loans.

The model prevents Ponzi games in the sense that individuals cannot borrow without bound to finance their unobservable (private) consumption. This is because, for any given publicly-observable capital stock of an entrepreneur, the intermediary's zero-profit condition (5) effectively places an upper bound on borrowing. We illustrate this point in Figure 1 which displays a typical relationship between l and r with fixed k under condition (5). We see that as an amount of borrowing increases, the loan rate will eventually be so high that the borrowing approaches a finite asymptote – our version of "credit rationing" whereby loan rates serve as a screening device. While the aggregate constraint (7) insures that ρ_t is free of risk, the individual equilibrium solution depends on the three state variables: w_t, ρ_t, and the distribution of final wealth. The distribution of wealth corresponds to that of entrepreneur population, as we distinguish entrepreneurs from each other according to final wealth. To see this we note first that given an initial capital k_0, final wealth w_t follows a first-order Markov process because current production relies only on the capital stock accumulated last period. By the law of large numbers with a continuum of i.i.d. random

variables (Judd [1985]), the probabilities over w_t's describe the distribution of entrepreneurs distinguished by w_t.

Throughout this paper we restrict our attention to the steady state equilibrium. The concept of steady state here is different from that of the deterministic steady state in a typical representative agent model. Here the steady state is stochastic in the sense that it concerns the stationary distribution of final wealth. One of the main features in our model of incomplete markets is that the distribution of wealth will become stationary as time t increases, whatever the initial distribution.[10] We have therefore the following definition.

DEFINITION 3.2. The (stochastic) steady state is the equilibrium in which the distribution of final wealth is independent of time t.

Although our heterogeneous agent model becomes a fiendish problem when the distribution of wealth is a state variable, the stationarity implies that the steady state distribution is no longer a state variable. In steady state, moreover, the bond rate ρ becomes constant. We can therefore solve the individual steady state problem by parameterizing individual choices c_t, k_t, b_t, l_t, and r_t as functions of the only one state variable "w_t". And if we let $\lambda_{1t}, \lambda_{2t}, \lambda_{3t}, \lambda_{4t}$, and λ_{5t} be Lagrangian multipliers for equations (6), (4), (5), $0 \leq b_t$, and $0 \leq l_t$ respectively, the Euler equation first-order conditions for the individual optimization are:

$$\lambda_{1t} = c_t^{-\gamma}, \tag{8}$$

$$\lambda_{1t} + \lambda_{3t}\frac{\partial j_t}{\partial b_t} = \beta\rho_t E_t[\psi(x_{t+1}) + (1-\psi(x_{t+1}))(1-\psi(y_{t+1}))]\lambda_{2t+1} + \lambda_{4t}, \tag{9}$$

$$\lambda_{3t}\frac{\partial j_t}{\partial l_t} + \beta r_t E_t[\psi(x_{t+1})\lambda_{2t+1}] = \lambda_{1t} + \lambda_{5t}, \tag{10}$$

$$\lambda_{3t}\frac{\partial j_t}{\partial r_t} + \beta l_t E_t\psi(x_{t+1})\lambda_{2t+1} = 0, \tag{11}$$

$$\lambda_{1t} + \lambda_{3t}\frac{\partial j_t}{\partial k_t} = \beta(A\alpha k_t^{\alpha-1} + \delta)E_t[\psi(x_{t+1}) + (1-\psi(x_{t+1}))(1-\psi(y_{t+1}))]\eta_{t+1}\lambda_{2t+1}, \tag{12}$$

and

$$\lambda_{1t} = \lambda_{2t}, \lambda_{4t}b_t = 0, \lambda_{5t}l_t = 0. \tag{13}$$

[10] The steady state is equivalent to the dynamic equilibrium if the initial condition for the model is the steady state solution.

To solve the above problem, we need to derive both the conditional distribution and the marginal distribution for w_t in the (time-invariant) steady state. We denote the conditional c.d.f. by "G" and the marginal c.d.f by "H". The functional for of G can be derived according to (4). Omitting the subscript t, we have the following form:

$$G(w|w_{-1}) = \begin{cases} F\left(\frac{w-\rho b_{-1}}{Ak_{-1}^{\alpha}+\delta k_{-1}}\right), & \text{if } w < \overline{w}; \\ F\left(\frac{w+r_{-1}l_{-1}-\rho b_{-1}}{Ak_{-1}^{\alpha}+\delta k_{-1}}\right), & \text{if } w \geq \overline{w}. \end{cases} \tag{14}$$

Recall that $F()$ is the c.d.f. on η_t. Since $G(\cdot|w_{-1})$ is a probability measure, there exits a unique marginal c.d.f. $H()$ such that the following Riemann-Stieltjes integral holds:

$$H(w) = \int_0^{\infty} G(w|w_{-1})dH(w_{-1}). \tag{15}$$

Properties of Riemann-Stieltjes integrals imply that $H(w)$ is discontinuous at $\overline{w}$ with positive probability when $l > 0$ (Lindgren [1976]). That is to say, a population of bankrupt entrepreneurs has a concentration at $w = \overline{w}$.

With $H(w)$ satisfying (15), social welfare in the steady state is measured by

$$U = \int_0^{\infty} u(c(w))dH(w)$$

Similarly, the aggregates of capital stock, bonds, and loans are

$$K = \int_0^{\infty} k(w)dH(w), B = \int_0^{\infty} b(w)dH(w), \text{and } L = \int_0^{\infty} l(w)dH(w).$$

The steady state equilibrium is then characterized as: (i) the decision rules $c(w)$, $k(w), l(w)$ and $b(w)$, describing individual optimal choices which satisfy the Euler equations; (ii) the pricing function $r(w)$, describing how the loan rate depends on the individual's own wealth; (iii) the risk-free rate ρ, endogenously determined so as to make the intermediary's budget constraining (7) hold; (iv) the steady state distribution of final wealth, $H(w)$, satisfying (15).

4. QUANTITATIVE RESULTS AND THEIR IMPLICATIONS

To calibrate the model, we use the minimum weighted residual method (Judd [1992]). We note that both the distribution of wealth w and the

risk-free rate ρ are endogenously determined in the steady state. We characterize the decision rules and the pricing function as a linear combination of a finite number of elements in a Banach space of continuous functions in the following forms:

$$\begin{aligned} c(w) &= \sum_{n=0}^{N-1} a_n^c T_n(w), \\ k(w) &= \sum_{n=0}^{N-1} a_n^k T_n(w), \\ r(w) &= \sum_{n=0}^{N-1} a_n^r T_n(w), \end{aligned}$$

$$l(w) = \sum_{n=0}^{N-1} a_n^l T_n(w),\ b(w) = \sum_{n=0}^{N-1} a_n^b T_n(w), \tag{16}$$

where N is an integer, T_n's continuous polynomial functions, and a_n's corresponding coefficients. Conditional expectations is the Euler equations (9) to (12) are evaluated by Gaussian integrations (Davis and Rabinowitz [1984]).[11]

We calibrate the model using the following parameter values– most of them commonly utilized in other intertemporal growth models: $A = 1$, $\alpha = 0.3$, $\beta = 0.9$, $\varsigma = 0.01$, $\delta = 0.9$, $\varkappa = 0.01$, and $l = 1$.[12] The distribution density function of η is with bounded support and has a triangular form: η for $0.001 \leq \eta < 1$ and $2 - \eta$ for $1 \leq \eta < 1.999$. These parameter values are also used to calibrate our autarkic case. Table 2 reports a number of results for the aggregate variables in both the autarkic equilibrium and the equilibrium with bankruptcy law.

The results indicate that social welfare in bankruptcy equilibria improves upon the autarkic equilibrium and that a moderate exemption level raises both an aggregate amount of borrowing and social welfare. The relationship between welfare and the exemption level, however, is non-monotone. From the intuition similar to that in our static model in Section 2, we note that too high an exemption level (e.g.,$\overline{w} = 0.5$) tends to force the intermediary's average return on loans below the risk-free rate. As consequence, the amount of loans will be reduced. Indeed, as the exemption level approaches infinity, the probability of bankruptcy for every borrower

[11] For the detail of our computational method, see Judd [1992] or the independent technical appendix.

[12] We stress that the implications and qualitative conclusions drawn in this paper hold when we use different parameter values.

TABLE 2.

Numerical Results for the Equilibrium

$\overline{w}$	U	K	L	ρ
Autarky	-0.54	0.93	N/A	N/A
0.13	-0.33	1.05	0.06	1.00
0.43	-0.27	1.12	0.22	1.02
0.50	-0.29	1.09	0.07	1.00

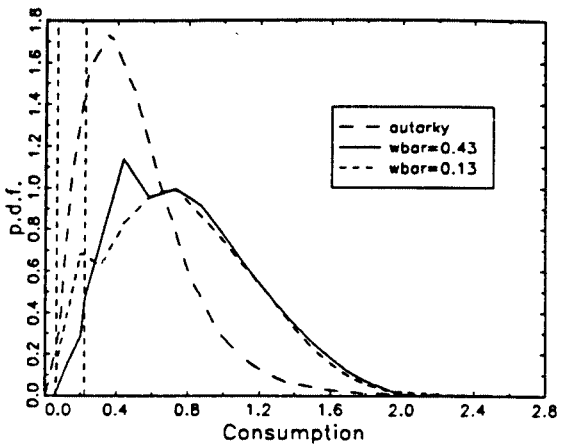
will become one and therefore no lending will take place. The resulting bankruptcy equilibrium becomes the autarkic one.

In short,our model suggests that only when the exemption level is moderate can the intervention be welfare enhancing. With the hindsight we gained in Section 2, this result may not be surprising when an upward adjustment to a moderate exemption raises the aggregate amount of capital as well (Table 2). But unlike the static model in Section 2, a positive relationship between total capital and total borrowing does not always hold in our capital accumulation model. In fact, there are situations in which the aggregate capital falls and the gross risk-free bond rate is below 1.00, while a moderate exemption both increases aggregate borrowing and improves social welfare.[13]

As noted before, our intertemporal model accentuates how the distributions of individual variables change in response to regulatory intervention in the form of a moderate exemption, because these redistributions underlie the aggregate behavior discussed above. Figures 2 and 3 display the distributions of consumption and capital in the autarkic case, in the economy with $\overline{w} = 0.13$, and in the economy with $\overline{w} = 0.43$. We note that the first vertical line in Figure 2 marks the amount consumed by the entrepreneurs with final wealth of 0.13 when $\overline{w} = 0.13$ and the second line by those with 0.43 when $\overline{w} = 0.43$. Similarly, the first vertical line in Figure 3 marks the capital stock accumulated by the entrepreneurs with wealth of 0.13 when

[13]One of the situations is when the distribution of η has instead unbounded support with, e.g., the following log-normal form of density: $\eta = 0.001 + \eta^*$ where $\log(\eta^*) \tilde{} N(m_1, \sigma_1^2)$ and $m_1 = \log(0.999) - \sigma_1^2/2$. In this case, the population concentrates heavily on entrepreneurs of meager wealth (the poor). Consequently, a significant amount of bonds invested by entrepreneurs of abundant wealth (the rich) is used to finance investment desired by the very poor who have little capital. Meanwhile, since the distribution of idiosyncratic shocks in this case has a large concentration (mode) around low values, entrepreneurs with large accumulation of capital are likely to receive extremely unfavorable shocks and thereby their wealth are subject to great uncertainty. As an equilibrium outcome, highly risk-averse rich entrepreneurs are willing to invest the optimal amount of wealth in bonds even though the bond rate is below 1.00. The intermediary in this case serves to provide insurance for the rich against disastrous shocks as well as to channel funds to the poor.

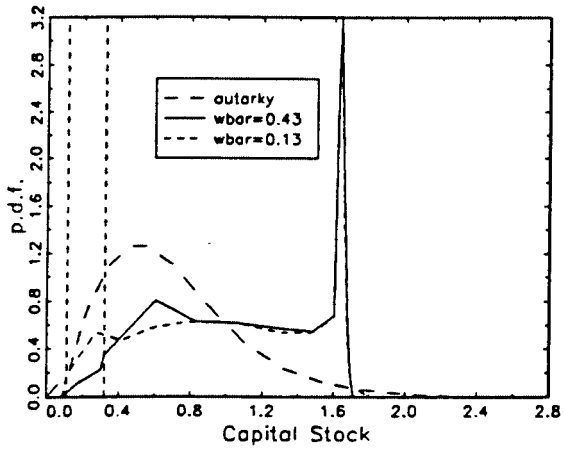
FIG. 2. Autarky and Bankruptcy



$\overline{w} = 0.13$ and the second line by those with 0.43 when $\overline{w} = 0.43$.[14] As these figures illustrate, when our model economy permits intermediation and allows intervention in the form of bankruptcy law, the distribution density functions of both consumption and capital shift to the right compared to those in the autarkic situation. In other words, in an economy with bankruptcy law a larger portion of population enjoys a higher level of both consumption and accumulated capital stock when compared to the autarkic situation. Let us look into the results of a moderate exemption (i.e., $\overline{w} = 0.43$) in Figures 2-4. A change of exemption, as one would expect, mainly affects the ability of poor entrepreneurs to borrow and consume. We thus see, in Figures 2 and 3, that shifts in the distributions of consumption and capital take place in the region of low values as the exemption level is adjusted from 0.13 to 0.43. There is little change in distribution for the high values of consumption and capital. In an economy with bankruptcy law, a moderate exemption ($\overline{w} = 0.43$) allows the poor to borrow more and consequently many of them end up enjoying higher consumption than those in the economy with $\overline{w} = 0.13$ – a phenomenon reflected by the first peak of the density function with $\overline{w} = 0.13$ and by the first peak with $\overline{w} = 0.43$ (Figure 2). Meanwhile, the risks of bankruptcy rise across all borrowers (and do so very sharply for the extremely poor)

[14]The population of entrepreneurs concentrates at $\overline{w} = 0.13$ with the probability of 0.005, and at $\overline{w} = 0.43$ with 0.013.

FIG. 3. Autarky and Bankruptcy

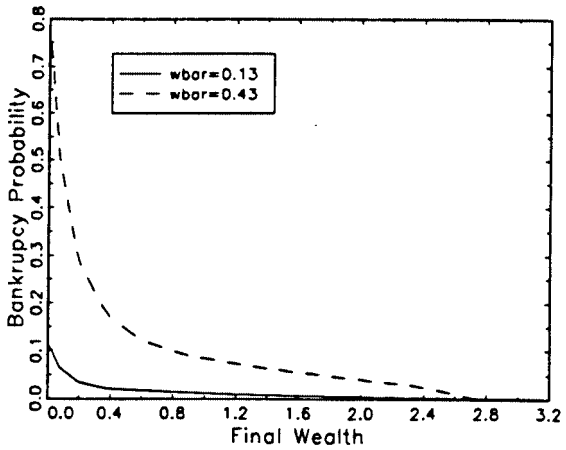


when $\overline{w}$ increases from 0.13 to 0.43 (Figure 4).[15] For those who end up being in the state of bankruptcy, they consume an amount at and below the threshold level marked by the first vertical line when $\overline{w} = 0.13$ or by the second vertical line when $\overline{w} = 0.43$ (Figure 2). As for capital stock, the accumulation rises with wealth but only to a certain point beyond which the rich invest the rest of their wealth in bonds to earn a higher return. We thus see in Figure 3 the second peak of the density function around the high level of capital stock. When we compare the results in bankruptcy equilibria with the autarkic results,the distribution densities of consumption and capital shift to the right across all entrepreneurs (Figures 2 and 3). In an economy with bankruptcy law, Figure 2-4 show that while permitting debts to be riskier especially for the very poor, and upward adjustment to a moderate exemption enables more borrowers to enjoy higher consumption and to accumulate more capital stock – the distribution densities shift to the right in the region of low values. At the same time, the distribution of consumption and capital among the rich change little. We view these cross-agent results as distributive improvement and this kind of intervention as a desirable one, for the usual trade-off between equity and efficiency disappears.

Such findings on distributive equity can be best summarized by the average time it takes, owing to idiosyncratic shocks, for a wealthy entrepreneur

[15] Of course, the state of bankruptcy in the economy with $\overline{w} = 0.13$ is different from that with $\overline{w} = 0.43$. Entrepreneurs with the wealth of 0.30, for example, are not in the state of bankruptcy when $\overline{w} = 0.13$, but they are when $\overline{w} = 0.43$.

FIG. 4. Bankruptcy Case



to become poor as well as for a poor entrepreneur to become rich. For this purpose, we divide final wealth into six categories on the scale of 1 to 6 whereby 1 classifies wealth below 0.37, 2 between 0.37 and 0.88, 3 between 0.88 and 1.57, 4 between 1.57 and 2.35, 5 between 2.35 and 3.53, and 6 above 3.53. To obtain an average transitional time, we conduct Monte Carlo simulations with 4,000 repetitions. The results change hardly at all when the number of repetitions is further increased.

Our computed results are: for the poor (category 1) to become rich (category 6), it takes on average 56 periods in the autarkic economy (Figure 5.1), 18 periods in the economy with $\overline{w} = 0.13$ (Figure 5.2), and 17 periods with $\overline{w} = 0.43$ (Figure 5.3). The transition takes one third as long in an economy with bankruptcy law as in the autarkic economy, implying that intervention in the form of bankruptcy law improves distributive equity by allowing the poor to be wealthy at a significantly faster speed. The transition speeds up little when $\overline{w}$ is adjusted up from 0.13 to 0.43. This is because an upward adjustment in moderate exemptions mainly affects the distribution of less wealthy population. In order to see how such adjustment improves distributive equity in a bankruptcy economy, let us examine the average time changing from *rich to poor*.[16] We see from Figure 5.5 and 5.6 that the transition is substantially prolonged from 49 periods with $\overline{w} = 0.13$ to 91 periods with $\overline{w} = 0.43$. When $\overline{w} = 0.43$, the borrower's

[16] As one expects, the transition from rich to poor in the situation of autarky (Figure 5.4) is much rapider than in an economy with bankruptcy law (Figures 5.5 and 5.6), with only 29 periods on average.

FIG. 5.1. Typical Autarkic Case

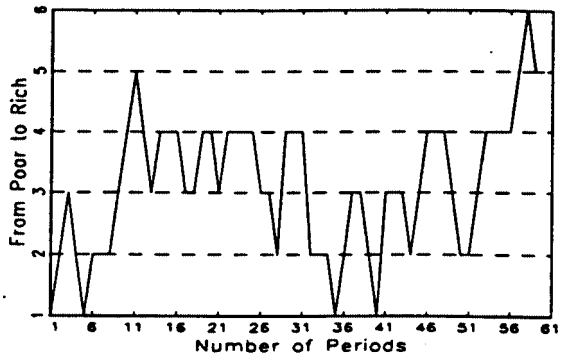


FIG. 5.2. Typical Bankruptcy Case (wbar= 0.13)

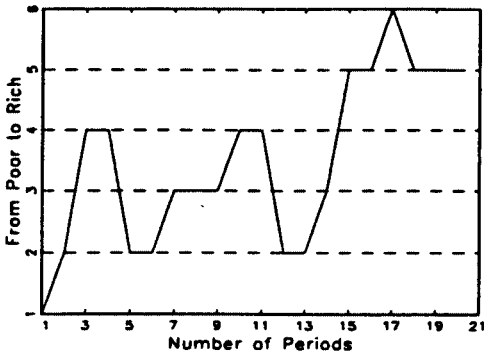


FIG. 5.3. Typical Bankruptcy Case (wbar= 0.43)

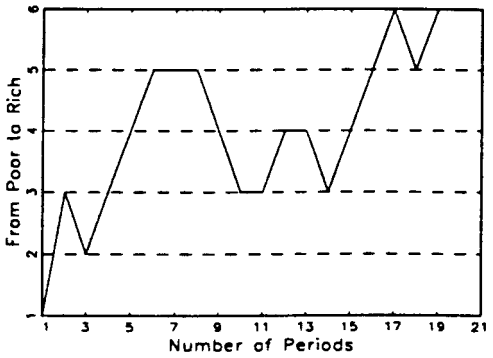


FIG. 5.4. Typical Autarkic Case

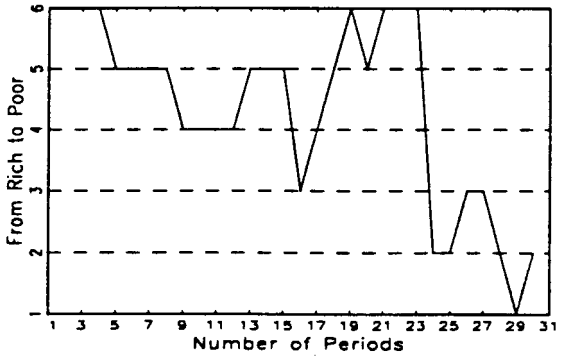


FIG. 5.5. Typical Banktuptcy Case (wbar= 0.13)

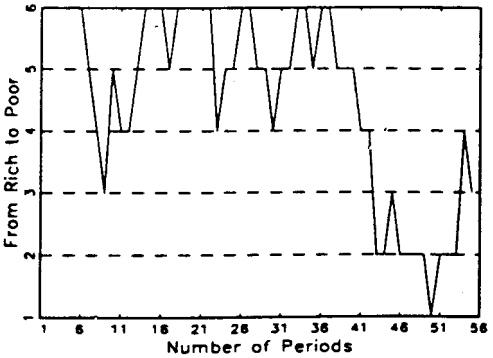
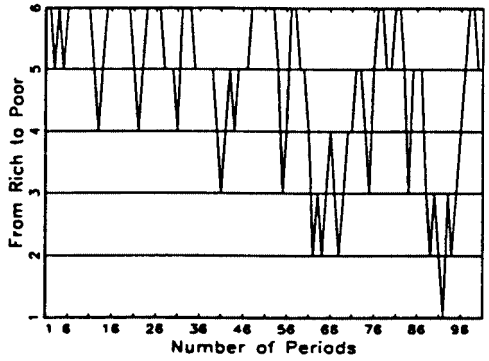


FIG. 5.6. Typical Bankruptcy Case (wbar= 0.43)



assets can be exempted up to 0.43 in the state of bankruptcy, thus giving the borrower some protection from becoming poor (note, by "poor" here we mean final wealth of less than 0.37). When $\overline{w} = 0.13$, however, all the borrowers in the state of bankruptcy are already in the poor category (category 1), thus making the transition from rich to poor much faster. An upward adjustment to a moderate exemption therefore tends to render entrepreneurs an easier access to loan markets and to protect them from being poor; and in this sense it enhances distributive equity.

5. CONCLUDING REMARKS

Although the role for bankruptcy law has been recently examined in the static security models of Dubey, Geanakoplos and Shubik [1995] and Zame [1993], it has been largely unexplored in intertemporal models with capital accumulation. In the spirit of these previous works, we introduce bankruptcy in an intertemporal model, and discuss how bankruptcy law is specified in an environment of incomplete markets with idiosyncratic shocks and capital accumulation. We explore the role for regulatory intervention when financial markets are incomplete by examining how an adjustment in the exemption level redistributes individual wealth and influences social welfare. The present model has a clear result: intervention of this sort can improve both social welfare and distributive equity. For theoretical pith, this result highlights the regulatory role in promoting equity and efficiency in an environment of incomplete markets. For practical use, it may help us understand the effects on the actual economy of the recent U.S. personal bankruptcy reform embodied in the U.S. Bankruptcy Code of 1978.

Indeed, the bankruptcy law in our model is specified so as to reflect certain institutional aspects in the U.S. economy. One may think of our specification as an approximation to the "straight bankruptcy" proceeding in Chapter 7 of the U.S. Bankruptcy Code.[17] In a theoretical model with complete asset markets, default penalties can be made extremely harsh so that bankruptcy disappears (Dubey and Shubik [1979]). But when asset markets in the actual economy are incomplete, harsh punishment on bankruptcy may become socially undesirable (e.g., Luckett [1988], Dubey, Geanakoplos and Shubik [1995]). The spirit of modern bankruptcy law in some actual economies is to develop straightforward institutional rules

[17] "Straight bankruptcy", the most commonly used proceeding in the court, pertains to the exemption of a bankrupt person's assets and the liquidation of her or his estate. The percentage of straight bankruptcy cases in all bankruptcy cases during the postwar period has on average been about 75. Moreover, voluntary bankruptcy cases have been an extremely high percentage in all straight bankruptcy cases. For the institutional detail, see *Annual Report of the Director, Administrative Office of the United States Courts, various issues.*

under which some of debtors' assets ban be protected in the he state of bankruptcy.

Our model follows a strand of the finance and economics literature which focuses on entrepreneurial firms and debt contracts. It therefore abstracts from other details - notably corporate capital structure and the related corporate bankruptcy law. Although such abstraction enables us to gain clear and intuitive results here, we think of the exercise in this paper as an analytic step towards models of incomplete markets that capture more features that are important in understanding the role of government intervention in business fluctuations and economic growth. In particular, it is argued that bankruptcy rates, as well as bankruptcy risks (or some measures of them), play a structural role for the transmission of monetary policy in some actual economies (e.g., Bernanke [1981, 1983], Calomiris and Hubbard [1989], Sims and Zha [1994]). The present model can be extended to analyze the dynamic transmission mechanisms of government policies and regulations. One possible extension is to include aggregate uncertainty in a dynamic model, although such inclusion makes the problem technically fiendish (because the distribution of wealth becomes a state variable). It is our hope, therefore, that the theoretical contribution here will be useful to future study pertaining to government policies and regulations.

REFERENCES

Aghion, P., O. Hart, and J. Moore, 1992, The economics of bankruptcy reform. *NBER working Paper #4097.*

Baumol, W. J., 1986, *Superfairness.* The MIT Press, Cambridge, Massachusetts.

Bernanke, B. S., 1981, Bankruptcy, liquidity, and recession, and recession. *American Economic Review Papers and Proceedings* **71**, 155-159.

Bernanke, B. S., 1983, Nonmonetary effects of the financial crisis in the propagation of the great depression. *American Economic Review* **73**, 257-276.

Blanchard, O. J. and S. Fischer, 1989. *Lectures on Macroeconomics.* The MIT Press, Cambridge, Massachusetts.

Boyes, W. J. and R.L. Faith, 1986, Some effects of the bankruptcy reform act of 1978. *Journal of Law and Economics* **XXIX**, 139-149.

Calomiris, C. W. and R. G. Hubbard, 1989, Prices flexibility, credit availability, and economic fluctuations: Evidence from the united states, 1894-1909. *Quarterly Journal of Economics* **104**, 429-452.

Calomiris, C. W. and R. G. Hubbard, 1990, Firm heterogeneity, internal finance, and 'credit rationing'. *Economic Journal* **100**, 90-104.

Clower, R. W., 1967, A reconsideration of the microfoundations of money. *Western Economic Journal* **6**, 1-9.

Davis, P. J. and P.Rabinowitz, 1984, *Methods of Numerical Integration*, second Edition. Academic Press.

De Meza, D. and D. C. Webb, 1987, Too much investment: A problem of asymmetric information. *Quarterly Journal of Economics* **102**, 281-292.

Dubey, P. and M. Shubik, 1979, Bankruptcy and optimality in a closed trading mass economy modeled as a noncooperative game. *Journal of Mathematical Economics* **6**, 115-134.

Dubey, P., J. Geanakoplos, and M. Shubik, 1995, Default and efficiency in a general equilibrium model with incomplete markets. *Manuscript*.

Gale, D. and M. Hellwig, 1985, Incentive-compatible debt contracts: The one-period problem. *Review of Economic Studies* **LII**, 647-663.

Hart, O. and J.Moore, 1994, A theory of debt based on the inalienability of human capital. *Quarterly Journal of Economics* **439**, 841-879.

Hellwig, M. F., 1977, A model of borrowing and lending with bankruptcy. *Econometrica* **45**,1879-1906.

Judd, K. L., 1985, The law of large numbers with a continuum of IID random variables. *Journal of Economic Theory* **35**, 19-25.

Judd, K. L, 1992, Minium wighted residual methods for solving dynamic economic models. *Journal of Economic Theory* **58**, 410-452.

Lindgren, B. W., 1976, *Statistical Theory*. MacMillan Publishing Co., Inc., New York.

Luckett, C. A., 1988, Personal bankruptcies. *Federal Reserve Bulletic* **74(9)**, September.

Moore, R. R., 1993, Asymmetric information, repeated lending, and capital structure. *Journal of Money, Credit, and Banking* **25**, 393-409.

Shepard, L., 1984, Personal failures and the bankruptcy reform act of 1978. *Journal of Law and Economics* **27**, 419-39.

Sims, C. A. and T. Zha, 1994, Does monetary policy generate recessions?: Using less aggregated price data to identify monetary policy. *Manuscript*.

Smith, V. L., 1972, Devaulf risk, scale, and the homemade leverage theorem. *American Economic Review* **62**, 66-76.

Zajac, E. E. 1986, Perceived economic justice: The example of public utility regulation. In:*Cost allocation: Methods, principles, applications*, ed. by H. P. Young. North Holland, Amsterdam

Zame, W. R, 1993, Efficiency and the role of default when security markets are incomplete. *American Economic Review* **83**, 1142-1164.