Liquidity, Bargaining, and Multiple Equilibria in a Search Monetary Model[*]

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In this paper I construct a search monetary model with capital accumulation where money and goods are both divisible. Agents in matches determine the terms of trade through a sequential bargaining process and they face trading restrictions that require the quantity of money traded not to exceed what the buyer brings into the match. I show that sellers' share of the match surplus decreases with the severity of the trading restrictions. Such endogenous surplus shares generate multiple, self-fulfilling monetary steady states. When liquidity is interpreted as the number of transactions, the steady state with higher aggregate activities has higher liquidity. In both steady states, an increase in the money growth rate increases aggregate output and consumption by increasing the number of matches. © 2001 Peking University Press

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1. INTRODUCTION

Real aggregate output and the money stock are positively correlated in post-war U.S. data (e.g., Sims, 1992, and Cooley and Hansen, 1995). This empirical correlation, often appearing as the Phillips curve, is a building block of many reduced-form models that are popular for policy analysis (e.g., Taylor, 1993). In the literature there are two classes of models that capture this empirical correlation with utility-maximizing households. One is sticky price models (e.g. Cooley and Hansen, 1995, section 5), where nominal prices or wages are sticky and so positive monetary shocks stim-

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ulate real output by increasing effective aggregate demand. Another class is limited participation models (e.g., Lucas, 1990), where only a fraction of agents receive monetary injections and such injections increase real output by increasing "liquidity" in the economy. In such models, liquidity is constrained by a cash-in-advance restriction in the goods market or the labor market.[1] In this paper, I construct a model that uses search frictions to generate real effects of monetary shocks without nominal stickiness or limited participation.

The motivation for constructing a new model is that neither of the two existing classes of models is satisfactory. Models with sticky prices or wages lack a rigorous foundation for why there is nominal rigidity. In addition, they do not generate sufficient propagation for monetary shocks given realistic duration of the nominal stickiness (Chari, et al. 1998). Limited participation models offer a more robust explanation for the real effects of monetary shocks but they, too, have serious shortcomings. They do not carefully spell out the physical environment which gives rise to the cash-in-advance constraint and they do not generate sufficient propagation for monetary shocks.

In this paper I extend an earlier model (Shi, 1999) that features costly search in the goods market. As in a typical search model (e.g., Shi, 1995, and Trejos and Wright, 1995), agents are matched randomly and bilaterally, and they bargain over the terms of trade sequentially. Preferences and production technologies are such that make it difficult for two agents to have a double-coincidence of wants in barter. Agents use fiat money as a medium of exchange and this gives rise to trading restrictions on money that resemble a cash-in-advance constraint. Unlike limited participation models, however, all households receive money injections and do so before exchanges take place in the period. Monetary injections affect real output by affecting the fraction of time that each household spends in producing versus shopping.

More specifically, the search model here has divisible money and goods, and it allows for capital accumulation. Following my previous work (Shi, 1997, 1998, 1999), I model each household as a collection of a large number of members who share the matching risks. This modelling device eliminates the analytically intractable, non-degenerate distributions of consumption and capital stocks across households that would otherwise arise in the random matching environment. I allow each household to choose consumption, capital accumulation, money balances, the fraction of producers (sellers), and the terms of trade to propose in matches. Capital is modelled as con-

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[1] A common restriction in limited participation models is that firms must pay workers with cash before production. Monetary injections increase banks' loanable funds, which allow firms to pay wages to more workers, thus increasing real output.

sumption goods that are set aside from current consumption, as in a typical neoclassical growth model.

A buyer and a seller in a match bargain over the terms of trade in a sequential fashion, where the buyer is randomly chosen by nature to be the proposer with probability $\theta \in (0, 1)$ in each round. Since two agents in a match are separated from other agents, a trade is constrained by the amount of money the buyer brings into the match. Both buyers and sellers face such a trading restriction on money when they propose. The trading restrictions bind when the gross rate of money growth exceeds the discount factor.

The critical feature of the model is that the shares of surplus which buyers and sellers get in desirable matches depend on the degree to which the trading restrictions on money bind. Because a seller in a match has a positive probability $(1-\theta)$ to be the proposer, he shares a part of the cost of the trading restrictions and so he gets less than $(1 - \theta)$ share of the match surplus. In fact, the larger the shadow price of the trading restrictions, the smaller share of the match surplus the seller gets. Through the shadow price of the trading restrictions on money, money growth affects the seller's share of the match surplus, which in turn affects each household's choice of the fraction of sellers and changes real output.

There are multiple, self-fulfilling monetary steady states because agents' shares of the match surplus are endogenous. If households expect that the trading restrictions will not bind severely, then sellers' share of the match surplus will be high. Anticipating this surplus division, households allocate more members (more time) to be sellers. This increases the aggregate number of matches and increases aggregate supply of goods. Thus, the purchasing power of money rises, which fulfills the expectations that the trading restrictions on money will not bind severely. In this steady state, aggregate consumption and output are high. On the other hand, if households expect that the trading restrictions on money will bind severely, then sellers' share of the match surplus will be low and households will reduce the number of sellers. This reduces the number of matches and reduces aggregate supply of goods, which leads to a low purchasing power of money and fulfills the expectations of severely binding trading restrictions. In this steady state, aggregate consumption and output are low. When liquidity is interpreted as the number of transactions, there is a positive relationship between liquidity and aggregate activities across steady states.

An increase in the money growth rate increases aggregate consumption and output in both steady states. This is because an increase in the money growth rate creates a positive extensive effect, i.e., it increases the number of matches in both steady states. In contrast to this extensive effect, money growth has opposite intensive effects in the two steady states, i.e., the effects on the quantities of trade in each match. In the low-activity steady

state, an increase in money growth increases the quantities of goods traded in each match and reduces the degree to which the trading restrictions on money bind. These intensive effects reinforce the extensive effect to increase aggregate output. In the high-activity steady state, an increase in money growth reduces the quantities of goods traded in each match and increases the degree to which the trading restrictions on money bind. However, these negative intensive effects are dominated by the positive extensive effect.

The non-Walrasian nature of the model is important for the main results. First, money growth affects liquidity by affecting sellers' share of match surplus in sequential bargaining, which is a non-Walrasian characteristic. Second, when households change their number of sellers in response to the change in sellers' surplus share, aggregate supply of goods changes because the number of successful matches increases with the number of sellers. This is also a non-Walrasian characteristic. In a Walrasian model, in contrast, trades take place instantaneously and so the number of matches is irrelevant for equilibrium; instead, money growth affects equilibrium exclusively through its intensive effects. By construction, my model does not rely on nominal rigidity or limited participation to deliver real effects of money growth. In particular, all households in this model receive the monetary transfer before they exchange, and so participation is not limited to only a fraction of agents.[2]

This paper closely follows my previous work (Shi, 1999). The framework contrasts with most search models of money in two dimensions. First, there is capital accumulation. This allows me to analyze the relationship between money and growth, a relationship that was emphasized by Johnson (1962) and Sidrauski (1967). Second, both money and goods are divisible here. In contrast, most search models of money restrict money to be indivisible (e.g., Shi, 1995, and Trejos and Wright, 1995), or goods to be indivisible (Green and Zhou, 1998), or both money and goods to be indivisible (Li, 1994). Those models are incapable of capturing the effects of money growth. When money is indivisible it is impossible to allow for money growth while maintaining a positive value for money. When goods are indivisible it is impossible to examine how producers change the quantity of production to respond to money growth. To allow both money and goods to be divisible, however, one encounters non-degenerate distributions of money holdings, consumption, and capital stock across agents, which are notoriously intractable analytically. In previous attempts (Shi, 1997, 1998, 1999), I introduced the device of large households to smooth the idiosyncratic matching risks within each household. This device allowed me to characterize symmetric monetary equilibria using a representative house-

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[2]Moreover, the monetary propagation mechanism is likely to be stronger in a search model than in a limited participation model, as is shown previously in a similar model (see Shi, 1998).

hold and to focus on the aggregate effects of monetary policies. The current paper uses this device.

In the previous work (Shi, 1999), I assumed that the number of sellers is exogenous and that buyers make a take-it-or-leave-it offer. In the current paper I endogenize the fraction of sellers and allow both sellers and buyers to have positive shares of the match surplus. Although these extensions significantly complicate the analysis of the terms of trade, they allow me to generate the interesting result of multiple equilibria. They also allow me to show that the positive relationship between money growth and aggregate output can be robust across equilibria.

2. A DESCRIPTION OF THE ECONOMY

Consider a discrete-time economy with H types of households and H types of goods, where $H \geq 3$. There are a large number of households in each type, whose size is normalized to one. I will refer to one arbitrary household of type $h \in H$ as household h and use lower-case letters to denote its decisions. Capital-case variables denote other households' decisions, which are taken as given by household h. Of course, lower-case variables are equal to the corresponding capital-case variables in all symmetric equilibria.

Different types of households differ in their preferences and production capabilities. A household h consumes only good h, which is called the household's consumption good, and produces only good $h+1$ (with modulus H). This implies that two different households can never have double coincidence of wants in barter, i.e., they can never both supply consumption goods to each other. In this environment, money might be valued as a medium of exchange. To capture this transactions role of money, I follow Kiyotaki and Wright (1993) to assume that agents are randomly matched in pairs in each period and that agents' transaction histories are private information.[3] Normalize the matching rate to 1 and let $\alpha \equiv 1/H$ denote the probability with which the agent's partner is (any) one particular type.

Money is an intrinsically useless object, which means that money yields no direct utility or productive services. Also, money is perfectly storable. Monetary exchanges are possible for household h when the household meets household $(h - 1)$, in which case household h exchanges money for goods, or when it meets household $(h + 1)$, in which case household h produces goods in exchange for money. I call these two types of matches the *desirable matches* of household h, in which there is a single coincidence of wants

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[3] When trading histories are public, there can be credit arrangements or gift-giving exchanges that are supported by trigger strategies (see Kocherlakota, 1998). Similarly, bilateral matching excludes possible trade arrangements among three or more types of households.

in goods between the two households and money serves as a medium of exchange.

Except the above differences in preferences and production capabilities, households are identical in other aspects. All households discount future with a discount factor $\beta \in (0, 1)$ and the intertemporal utility function is

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \varphi - (b + n_t)(\varphi - 1) - \alpha b n_t \Phi(l_t) \right]. \tag{1}$$

Here $u(c_t)$ is the household's utility of consumption, which is twice differentiable, strictly increasing and concave with $u'(0) = \infty$ and $u'(\infty) = 0$. The constant φ is the utility of leisure when an agent does not participate in the market, where $\varphi > 1$. If an agent participates in the market but does not produce, the utility of leisure is 1, and so $(\varphi - 1)$ measures the disutility of market participation. If an agent participates in the market and inputs l_t units of labor to produce goods, the utility of leisure is further reduced to $1 - \Phi(l_t)$. The household spends $(b + n_t)$ fraction of the time in the market and the chance to produce is $\alpha b n_t$, which I will explain later in detail. For simplicity, I assume that the cost function $\Phi(.)$ has the following form:

$$\Phi(l) = \Phi_0 l^\sigma, \qquad \sigma > 1, \ \Phi_0 \in (0, 1). \tag{2}$$

Production entails labor and capital. To allow for capital, I assume that consumers can store goods but producers cannot. A household can set aside a part of its consumption goods as productive capital. This setup ensures that consumption goods and capital are physically the same for a household, just as in a standard one-sector neoclassical growth model. In contrast to a standard growth model, however, a household's output and consumption good are not the same – one's output is someone else's consumption good. Since non-consumption goods are perishable, no one will exchange for them and so goods including capital cannot be used as a medium of exchange, ruling out commodity money.[4] Without loss of generality, I assume that capital does not depreciate.

The production function is as follows:

$$q = F(l, k/n) = F_0^{-\varepsilon} l^\varepsilon (k/n)^{1-\varepsilon}, \ \varepsilon \in (0, 1),$$

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[4] Allowing producers to be able to store their products introduces the possibility of commodity money and complicates the analysis considerably. Barring the possibility of commodity money, the existence of inventory will unlikely change the main qualitative results of the paper. As shown in Shi (1998), the existence of inventory does not eliminate the extensive effect of money growth. On the contrary, inventory can provide an additional propagation channel for money growth shocks to generate persistent, hump-shaped output responses.

where q denotes the quantity of output, l the labor input, (k/n) the capital input, and F_0 a positive constant. Given capital k/n and the quantity of output q, the required labor input is

$$l = \frac{k}{n} f\left(\frac{nq}{k}\right), \text{ where } f\left(\frac{nq}{k}\right) \equiv F_0 \left(\frac{nq}{k}\right)^{1/\varepsilon}. \tag{3}$$

Since capital and labor are complementary, a higher capital stock saves the labor input in production. Moreover, the function f defined above satisfies:

$$f(0) = 0, \ f'(\cdot) > 0, \ f''(\cdot) > 0 \text{ and } \frac{nq}{k} f' = \frac{f}{\varepsilon}. \tag{4}$$

A household chooses consumption, capital accumulation, labor input in production and the quantities of trade. As discussed in the introduction, these decisions are quite complicated in the random-matching framework when money and goods are both divisible, because there are non-degenerate distributions across agents over consumption, capital/money stocks and labor input.[5] For tractability I assume that each household consists of a continuum of members, who carry out different tasks but all share the same consumption and regard the household's utility as the common objective. Although each member's matching outcome is random, his consumption does not depend on his own luck, since idiosyncratic risks are smoothed within each household. I can then examine a representative household's decisions and focus on symmetric equilibria.[6]

Each household consists of money holders (buyers), producers (sellers) and leisure-seekers, each performing one task at a time. A money holder tries to exchange money for consumption goods and a producer tries to produce goods for money. A leisure-seeker enjoys leisure and does not participate in the market. Normalize the size of the household to 1. Let b be the fraction of buyers in the household, n_t the fraction of sellers, and $(1 - b - n_t)$ the fraction of leisure-seekers. To examine how money growth affects production, I assume that the household can choose the fraction of sellers (n_t) in each period t. The fraction of buyers, however, is fixed for simplicity (see Shi, 1999, for an analysis of an endogenous b).

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[5]Green and Zhou (1998) solve the distribution of money holdings in a model where goods are not divisible and capital accumulation is absent.

[6]Assuming risk-sharing is common in macroeconomic models (e.g., Lucas, 1990). At the aggregate level, the continuum in each household can be alternatively interpreted as a unit interval of time endowed to a representative agent in a standard macroeconomic model. Although this alternative interpretation is more natural, implementing it in the current setting is considerably more difficult. First, it is difficult to construct a matching technology that eliminates aggregate uncertainty throughout the trading period. Second, the sequential decisions inherited in the dynamic interpretation are more difficult to detail.

With this notation, I can clarify the meanings of the cost terms in the utility function (1). Because the household spends $(b + n_t)$ of its time in the market each period, the cost of market participation is $(b + n_t)(\varphi - 1)$. Among the sellers, only those who have desirable matches produce. Recall that, for a seller in household h, a desirable match is one in which the trading partner is a buyer from household $(h + 1)$. Since a seller meets an agent from household $(h + 1)$ with probability α and that agent is a buyer with probability b, a seller in household h has a desirable matches with probability αb. Because there are n_t number of sellers in household h, the total number of sellers in household h who produce in period t is $\alpha b n_t$. The total cost of production is $\alpha b n_t \Phi(l_t)$, as in (1).[7]

The household functions as follows. At the beginning of each period t, the household has k_t units of capital and m_t units of money. The household chooses the fraction of sellers, n_t, a consumption level, c_t, total money holdings for the next period, m_{t+1}, and total capital stock for the next period, k_{t+1}. The household also prescribes the trading strategies for its members. Then the household evenly divides the money stock to its buyers and capital to its sellers, each buyer having a money balance m_t/b and each seller a capital stock k_t/n_t. After the allocation the agents are randomly matched and agents in desirable matches carry out the prescribed trading strategies. After exchange members bring back their receipts of goods and money, and each member consumes c_t units of goods. Then the household receives a lump-sum monetary transfer, τ_t, and carries the stocks (m_{t+1}, k_{t+1}) to $t + 1$.

Let a household's value function from period t onward be $v(k_t, m_t)$.[8] For future use, I define the marginal value of money in period $t+1$, discounted to period t, as:

$$\omega_t \equiv \beta v_m(k_{t+1}, m_{t+1}). \tag{5}$$

Similarly, other households' marginal value of future money is Ω_t. To write the Bellman equation that defines the value function, I need to describe the terms of trade first.

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[7]Strictly speaking, if each agent meets exactly another agent in a period, the probability with which a seller meets a desirable buyer should be $\alpha b/(b + N_t)$. Simplifying it to αb does not change the qualitative results much, because the simplification preserves the property that sellers' matching rate increases with the number of buyers in exchange, which is the one important for the extensive effect. The probability with which a buyer meets a desirable seller is simplified in a similar way.

[8]I suppress the dependence of the value function on aggregate state variables, such as the money growth rate, the aggregate money stock and the aggregate capital stock.

3. BARGAINING AND HOUSEHOLD'S DECISIONS

3.1. Bargaining and Terms of Trade

When a seller and a buyer meet in a desirable match, they bargain over the quantities of money and goods to be traded, according to the instructions prescribed by their own households. Bargaining proceeds in the following sequential fashion adapted from Rubinstein and Wolinsky (1985). Immediately after being matched, one of the two agents is chosen by nature to propose, with the buyer being chosen with probability $\theta \in (0, 1)$ and the seller with $1 - \theta$. If the proposal is accepted, trade takes place immediately. If the proposal is rejected, a time interval Δ is passed and the negotiation proceeds to the next round, where the sequence of moves by nature and the agents repeats. In this paper I focus on the limit case $\Delta \to 0$ but, to characterize the limit outcome, I analyze the case $\Delta > 0$ first.[9]

For money to have a positive value in equilibrium, buyers must get a positive fraction of the match surplus. This requires that buyers be chosen as the proposer with positive probability, i,e., θ be bounded below. Also, for a household to choose a positive fraction of members to be sellers in equilibrium, sellers must also get a positive fraction of the match surplus, i.e., θ must be bounded above. The following assumption specifies these restrictions.[10]

ASSUMPTION 1. *The parameter θ satisfies*

$$\frac{\varepsilon(\sigma - 1)}{\sigma - \varepsilon} < \theta < \frac{1}{2 - \varepsilon}. \tag{6}$$

Let me select an arbitrary household and analyze its proposals in desirable matches. Since buyers and sellers can be the proposers with positive probability, a household must prescribe one proposal to its buyers and another proposal to its sellers. Let (q_t^b, x_t^b) be the instruction for a proposing buyer, where q_t^b is the quantity of goods that the buyer asks the seller to supply and x_t^b is the quantity of money that the buyer pays for the goods. The subscript b indicates that the proposer is a buyer. Similarly, let (q_t^s, x_t^s) be the instruction for a proposing seller. The proposals by other households' buyers are (Q_t^b, X_t^b) and (Q_t^s, X_t^s) by other households' sellers, which the particular household in discussion takes as given.

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[9]When $\Delta < 1$, discounting between bargaining rounds is not consistent with the discrete-time setup in my model. The purpose of introducing discounting within a period is to simplify the equilibrium analysis because it allows me to show that buyers' and sellers' proposals are equal to each other in the limit $\Delta \to 0$. If agents discount only between periods, the equilibrium is close to the one in this paper when β is sufficiently close to 1.

[10]Note that the interval for θ in the assumption is non-empty for all $\sigma > 1$ and $0 < \varepsilon < 1$.

For responding agents, the household prescribes a strategy $e^b \in \{0, 1\}$ for a buyer and $e^s \in \{0, 1\}$ for a seller, where $e = 1$ means "accepting the offer" and $e = 0$ means "rejecting the offer". When an agent rejects the partner's offer, he stays in the bargaining game. Let D_t^s be the expected surplus that a seller anticipates to get in the subgame after rejecting a buyer's proposal, discounted to the current round of bargaining with a discount factor β^Δ. Then $e_t^s = 1$ if the buyer's proposal gives the seller a surplus which is equal to or greater than D_t^s, and $e_t^s = 0$ otherwise. Similarly, let D_t^b be the expected surplus that a buyer anticipates to get in the subgame after rejecting a seller's proposal. Then $e_t^b = 1$ if the seller's proposal gives the buyer a surplus which is equal to or greater than D_t^b. In the current model a desirable match generates a positive surplus and so sequential bargaining yields immediate agreement.

Consider the proposal (q_t^b, x_t^b). The proposal must give the partner (the seller) an expected surplus which is equal to or greater than D_t^s; otherwise the partner will reject the proposal and proceed to the next round of negotiation. Thus,

$$\Omega_t x_t^b - \Phi \left(\frac{K_t}{N_t} f \left(\frac{N_t q_t^b}{K_t} \right) \right) \geq D_t^s, \tag{7}$$

where the left-hand side of the inequality is the seller's surplus from accepting the buyer's offer. In particular, Ω_t is the seller's household's marginal value of money and the argument of the function $\Phi(.)$ is the labor input needed to produce q_t^b units of goods. Rewrite the above inequality as follows:

$$x_t^b \geq \frac{1}{\Omega_t} \left[D_t^s + \Phi \left(\frac{K_t}{N_t} f \left(\frac{N_t q_t^b}{K_t} \right) \right) \right]. \tag{8}$$

This is a constraint to the household when it chooses the instructions for its proposing buyers.

There is another constraint to the household when choosing (q_t^b, x_t^b). That is, the household cannot instruct the buyer to offer more money than the buyer has:

$$x_t^b \leq m_t/b. \tag{9}$$

This constraint stems from the assumption that the household members cannot communicate with each other during the match. Because the trade must be executed on spot and each agent cannot borrow from other agents in the household, the agent complete the trade with whatever resource he has. [11] I call this constraint and the similar one for proposing sellers

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[11] The constraint (9) is a natural trading restriction on individual buyers if each agent behaves as an independent entity rather than belonging to a large household. In this

(described later) the *trading restrictions on money.* They will play a critical role for multiple equilibria.

To complete the description of the proposal (q_t^b, x_t^b), I compute the expected surplus D_t^s. After the seller rejects the buyer's offer, time elapses by Δ and nature chooses the proposer in the next round of negotiation. With probability θ nature chooses the buyer to be the proposer, in which case the seller's expected surplus is still D_t^s. With probability $(1-\theta)$, the seller will be chosen to be the proposer, in which case the seller proposes (Q_t^s, X_t^s). If the buyer accepts this proposal, the seller's surplus is characterized by the left-hand side of (7) with (Q_t^s, X_t^s) replacing (q_t^b, x_t^b). Thus, the expected surplus that the seller gets in the next round of bargaining, discounted to the current round, is

$$D_t^s = \beta^\Delta \left\{ \theta D_t^s + (1-\theta) \left[\Omega_t X_t^s - \Phi \left(\frac{K_t}{N_t} f \left(\frac{N_t Q_t^s}{K_t} \right) \right) \right] \right\}.$$

Re-arranging the equation I get

$$D_t^s = \frac{(1-\theta)\beta^\Delta}{1 - \theta \beta^\Delta} \left[\Omega_t X_t^s - \Phi \left(\frac{K_t}{N_t} f \left(\frac{N_t Q_t^s}{K_t} \right) \right) \right]. \tag{10}$$

Similarly, consider a desirable match between a seller of the household in discussion and another household's buyer. The buyer's expected surplus after rejecting the seller's offer is

$$D_t^b = \frac{\theta \beta^\Delta}{1 - (1-\theta)\beta^\Delta} \left[u'(C_t) Q_t^b - \Omega_t X_t^b \right]. \tag{11}$$

The seller's proposal (q_t^s, x_t^s) must satisfy $u'(C_t) q_t^s - \Omega_t x_t^s \geq D_t^b$, which can be rewritten as

$$x_t^s \leq \frac{1}{\Omega_t} \left[u'(C_t) q_t^s - D_t^b \right]. \tag{12}$$

Also, the seller cannot ask the buyer to give up more money than what the buyer has, i.e.,

$$x_t^s \leq M_t/b. \tag{13}$$

3.2. A Household's Decisions

In each period t, a household's decision variables are

$$a_t \equiv \left(c_t, n_t, k_{t+1}, m_{t+1}, (q_t^b, x_t^b), (q_t^s, x_t^s) \right).$$

sense, imposing (9) ensures that the construction of a large household does not change the fundamental frictions in the economy.

TABLE 1.

Agents and terms of trade in desirable matches

	I_{mt}: buyers in desirable matches		I_{pt}: sellers in desirable matches	
	propose	respond	propose	respond
set	I^p_{mt}	I^r_{mt}	I^p_{pt}	I^r_{pt}
measure	$\theta \alpha N_t b$	$(1-\theta)\alpha N_t b$	$(1-\theta)\alpha n_t b$	$\theta \alpha n_t b$
terms of trade	(q^b_t, x^b_t)	(Q^s_t, X^s_t)	(q^s_t, x^s_t)	(Q^b_t, X^b_t)

The quantities (q^b_t, x^b_t) are prescribed only for those buyers who meet desirable sellers and who are chosen to be the proposer; (q^s_t, x^s_t) are only for those sellers who meet desirable buyers and who are chosen to be the proposer. To clarify this distinction between proposing agents and responding agents, I use Table 1 to list the frequency of each type of match and the corresponding quantities of trade. The notation I indicates subsets of members in desirable matches, the superscript p on I proposing agents, the superscript r responding agents, the subscript m buyers and the subscript p sellers.

Taking other households' decisions as given, the household in discussion solves the following dynamic programming problem:

$$\text{PH} \quad v(k_t, m_t) = \max \left\{ u(c_t) + \varphi - (b+n_t)(\varphi - 1) - \theta \alpha b n_t \Phi \left(\frac{k_t}{n_t} f \left(\frac{n_t}{k_t} Q^b_t \right) \right) \right.$$
$$\left. -(1-\theta)\alpha b n_t \Phi \left(\frac{k_t}{n_t} f \left(\frac{n_t}{k_t} q^s_t \right) \right) + \beta v(k_{t+1}, m_{t+1}) \right\}$$

subject to

$$(8) \text{ and } (9) \text{ for every buyer in } I^p_{mt};$$

$$(12) \text{ and } (13) \text{ for every seller in } I^p_{pt};$$

$$c_t + k_{t+1} - k_t \leq \alpha b N_t \left[\theta q^b_t + (1-\theta) Q^s_t \right]; \qquad (14)$$

$$m_{t+1} \leq m_t + \tau_t + \alpha b n_t \left[(1-\theta) x^s_t + \theta X^b_t \right] - \alpha b N_t \left[\theta x^b_t + (1-\theta) X^s_t \right]; \quad (15)$$

$$c_t, k_{t+1}, m_{t+1} \geq 0 \text{ and } n_t \in [0, 1-b] \text{ for all } t.$$

The constraint (14) resembles a standard budget constraint in a neoclassical growth model, except that the right-hand side is not income (or output) but rather the receipts from exchange. The constraint (15) is the law of motion of the household's money holdings. In addition to receiving lump-sum monetary transfers (τ), the household obtains money when its sellers sell goods for money in desirable matches and pays out money when its buyers buy good with money. In such transactions, the quantity

of money received or paid depends in general on whether the household's member is a proposer or a respondent, as recorded by (15).

4. CHARACTERIZATION OF MONETARY EQUILIBRIA

The following is a definition of a symmetric monetary equilibrium:

DEFINITION 4.1.   A symmetric monetary equilibrium consists of each individual household's decisions $\{a_t\}_{t \geq 0}$, other households' decisions $\{A_t\}_{t \geq 0}$, and monetary transfers $\{\tau_t\}_{t \geq 0}$ such that the following requirements are met: (i) Given $\{A_t\}$ and $\{\tau_t\}$, a_t solves an individual household's maximization problem (PH); (ii) $a_t = A_t$ for all t; (iii) $0 < n_t < 1 - b$ and $0 < \omega_t m_t < \infty$ for all $t \geq 0$; and (iv) $\tau_t = m_{t+1} - m_t = (\gamma - 1)m_t$, where $\gamma > 0$.

Part (i) of the definition requires that each individual household's decisions be the best response to other households' choices. Part (ii) imposes symmetry across households. Part (iii) requires that the equilibrium be monetary. In particular, the restriction $n_t > 0$ requires that the measure of sellers in the economy be positive (otherwise money would have no value); the restriction $n_t < 1 - b$ requires that the measure of buyers be positive; and $\omega_t m_t > 0$ requires that money be valued. The restriction $\omega_t m_t < \infty$ is necessary for the household's maximization problem to have a non-trivial solution: If $\omega_t m_t = \infty$ for some t, the household can consume an infinite amount using money at t. Part (iv) specifies the monetary transfers and the gross rate of money growth (γ).

As stated earlier, I focus on symmetric monetary equilibria in the limit where the interval between two bargaining rounds (Δ) approaches 0. To characterize these equilibria, I start with the terms of trade. Let $\theta \alpha b N_t \lambda_t$ be the shadow price of the trading restriction on money, (9), which a proposing buyer faces. Let $(1-\theta)\alpha b n_t \pi_t$ be the shadow price of the trading restriction on money, (13), which a proposing seller faces. Both shadow prices are expressed in terms of period-t utility.[12] The following proposition describes the bargaining outcome in the limit case (the proof appears in Appendix 1):

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[12]Notice that the household does not face the cost of the trading restrictions when responding to an offer, because the response is either "Yes" or "No". The long expressions of the shadow prices of the trading restrictions simplify the expressions for equilibrium conditions. They result from the following normalization. Since (9) applies only to the subset of member in I^p_{mt} (i.e., those money holders who meet desirable sellers and who are chosen to be the proposer), the shadow price of (9) is multiplied by the measure of agents in the subset I^p_{mt}. Similarly, the shadow price of (13) is multiplied by the measure of agents in the subset I^p_{pt}.

PROPOSITION 1.   *When* $\Delta \to 0$, *there is immediate agreement between the two agents in a desirable match. All symmetric monetary equilibria satisfy* $(q_t^b, x_t^b) = (q_t^s, x_t^s) = (q_t, x_t)$ *and*

$$\omega_t x_t - \Phi_t = D_t^s = \frac{(1-\theta)\omega_t}{\omega_t + \theta\lambda_t}\left[u'(c_t)q_t - \Phi_t\right]; \qquad (16)$$

$$u'(c_t)q_t - \omega_t x_t = D_t^b = \frac{\theta(\omega_t + \lambda_t)}{\omega_t + \theta\lambda_t}\left[u'(c_t)q_t - \Phi_t\right]. \qquad (17)$$

$$\lambda_t/\omega_t = \frac{\pi_t}{\omega_t - \pi_t} = \frac{\varepsilon u'(c_t)q_t}{\sigma\Phi_t} - 1, \qquad (18)$$

where $\Phi_t \equiv \Phi\left(\frac{k_t}{n_t}f\left(\frac{n_t q_t}{k_t}\right)\right)$. *Therefore,* λ *and* π *are either both* 0 *or both positive.*

Proposition 1 characterizes a household's choices of the quantities of trade. This proposition illustrates a number of important features of a symmetric monetary equilibrium. First, the quantities of trade proposed by a buyer and a seller are the same when $\Delta \to 0$. This is a standard result in the sequential bargaining framework. If the seller's and the buyer's proposals were different from each other, at least one agent would gain by rejecting the partner's offer and proceed to the next round of bargaining, which is not optimal for the two agents.

Second, the trading restrictions on money, (9) and (13), are either both binding or both non-binding in symmetric equilibria. In fact, $\lambda = \omega\pi/(\omega - \pi)$. To explain this result, notice that a binding restriction on money increases the marginal cost of money to the proposing buyer because, to propose a larger quantity of goods, the buyer must give the seller a larger quantity of money in order to induce the seller to accept the offer, which makes the trading restriction (9) more binding. The shadow price λ_t measures this additional cost of money to the proposing buyer. To cover the additional cost, a buyer's proposal requires that the marginal utility of consumption exceed the cost of production by a fraction λ_t/ω_t, i.e., $u'(c_t) = (1 + \lambda_t/\omega_t)\Phi_t' f_t'$. Similarly, a binding restriction on money to the proposing seller, (13), reduces the marginal value of money to the seller because it restricts the amount of money that the buyer can give. The shadow price π_t measures this reduction in the value of money to the proposing seller. To compensate for this lower value of money, a seller's proposal requires that the marginal cost of production be less than the marginal utility of consumption by a fraction π_t/ω_t, i.e., $\Phi_t' f_t' = (1 - \pi_t/\omega_t)u'(c_t)$. For both the buyer's and seller's proposals to be optimal, it must be true

that $(1 + \lambda_t/\omega_t) = (1 - \pi_t/\omega_t)^{-1}$, which leads to $\lambda = \omega\pi/(\omega - \pi)$. If $\lambda > \omega\pi/(\omega - \pi)$, for example, the household can reduce the quantity of goods asked by proposing buyers and increase the quantity of goods supplied by proposing sellers. In this case, the utility gain from the relaxation of the buyer's trading restriction exceeds the increase in the cost of the seller's restriction. That is, the household has a net gain, which contradicts the optimality of the household's original proposals.

Third, buyers' and sellers' shares of the match surplus are endogenous and are different from those in a similar sequential bargaining framework without the trading restrictions on money. Without the trading restrictions on money, an agent's share of the match surplus is equal to the probability with which the agent is chosen to propose, i.e., $(1 - \theta)$ for a seller and θ for a buyer. With the trading restrictions, however, a seller gets a share $(1-\theta)\omega/(\omega+\theta\lambda)$ of the match surplus and a buyer gets the remainder (see (16) and (17)). Provided $\theta \neq 0$ or 1, these shares depend on how binding the trading restrictions on money are and hence they are endogenous. When the trading restrictions on money bind, a seller gets less than $(1 - \theta)$ share of the match surplus, because a seller has a positive probability to be the proposer and to face the trading restriction on money. A seller shares a part of the cost generated by the money restriction, which reduces the seller's surplus share. In fact, the seller's share decreases with λ/ω and the buyer's share increases with λ/ω. That is, the more binding the trading restrictions on money are, the lower the seller's share is and the higher the buyer's share is. As analyzed later, endogenous surplus shares generate multiple monetary equilibria.

Let me characterize the household's other choices. Combining the first-order conditions for c_t and k_{t+1} with the envelope conditions for k_t and m_t, I obtain the following conditions:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + \frac{\alpha b \sigma(1 - \varepsilon)}{\varepsilon u'(c_{t+1})} \cdot \frac{n_{t+1}}{k_{t+1}} \Phi\left(\frac{k_{t+1}}{n_{t+1}} f\left(\frac{n_{t+1}q_{t+1}}{k_{t+1}}\right)\right), \qquad (19)$$

$$\frac{\omega_t}{\beta\omega_{t+1}} = 1 + \theta\alpha n_{t+1}\frac{\lambda_{t+1}}{\omega_{t+1}}. \qquad (20)$$

The condition (19) characterizes the household's optimal trade-off between consumption and savings. As in a standard intertemporal model, the condition requires that the marginal rate of substitution between current and future consumption be equal to the rate of return to capital. The condition (20) requires that the capital loss to holding money be offset by money's role of relaxing the trading restrictions.

In any equilibrium with $n_t < 1 - b$, the first-order condition for n_t is:

$$\frac{\alpha b \Phi_t}{\varepsilon[1 + \theta \lambda_t/\omega_t]} \left\{ \sigma[1 - (2 - \varepsilon)\theta]\frac{\lambda_t}{\omega_t} - [(\sigma - \varepsilon)\theta - \varepsilon(\sigma - 1)] \right\} = \varphi - 1. \quad (21)$$

For $n_t > 0$, the left-hand side of (21) must be positive, which requires $\theta < 1/(2 - \varepsilon)$ and

$$\frac{\lambda_t}{\omega_t} > \frac{(\sigma - \varepsilon)\theta - \varepsilon(\sigma - 1)}{\sigma[1 - (2 - \varepsilon)\theta]} \equiv z_0. \quad (22)$$

I will focus on equilibria with $\lambda/\omega > 0$. To justify this focus, note that $\omega_{t+1} = \beta^{-1}\omega_t$ when $\lambda/\omega = 0$ (see (20)), i.e., the value of money grows over time at the discount rate. To satisfy the equilibrium requirement $\omega_t m_t < \infty$ in this case, money supply must shrink at the discount rate. Thus, for $\gamma > \beta$, a monetary equilibrium with $\lambda/\omega = 0$ does not exist. To ease exposition, I impose a stronger condition $z_0 > 0$, which is equivalent to the lower bound on θ in (6). Under this condition, $n > 0$ implies $\lambda/\omega > 0$ (see (22)). I will also focus on steady states, because dynamics are analytically difficult to describe.

5. MULTIPLE MONETARY STEADY STATES

5.1. Existence

A key variable in the equilibrium is

$$z_t = \lambda_t/\omega_t. \quad (23)$$

This variable measures how severely exchanges are constrained by money holdings. The larger z is, the more agents are constrained by money. As discussed before, I focus on equilibria with $\lambda_t > 0$, which is equivalent to $z_t > 0$ for all $t \geq 0$.

Use an asterisk to indicate steady state and define the following functions:

$$R(z) \equiv \frac{\varepsilon(\varphi - 1)}{\alpha b} \cdot \frac{1 + \theta z}{\sigma[1 - (2 - \varepsilon)\theta]z - (\sigma - \varepsilon)\theta + \varepsilon(\sigma - 1)}, \quad (24)$$

$$K(z) \equiv \Phi^{-1}\left(R(z)\right) \Big/ f\left(\frac{(\beta^{-1} - 1)(1 + z)}{\alpha b(1 - \varepsilon)}\right). \quad (25)$$

$$q(z) \equiv \frac{(\beta^{-1} - 1)(1 + z)}{\alpha b(1 - \varepsilon)} K(z). \quad (26)$$

The following proposition details the number of steady states (see Appendix 1 for a proof):

PROPOSITION 2.   *In the steady state, $\Phi^* = R(z^*)$, $k^*/n^* = K(z^*)$ and $q^* = q(z^*)$. The steady state value of z^* is given by the solution to the following equation:*

$$\frac{R(z)}{K(z)} = \frac{\varepsilon(\beta^{-1} - 1)}{\alpha b \sigma (1 - \varepsilon)} u' \left(\alpha b (\gamma \beta^{-1} - 1) \frac{q(z)}{z} \right). \tag{27}$$

There exists $\gamma_E > \beta$, where γ_E is defined in Appendix 1, such that at least one monetary steady state exists and satisfies $0 < n^* < 1 - b$. The number of such steady states is even if

$$\lim_{c \to 0} cu'(c) < \frac{\theta(\varphi - 1)(\gamma \beta^{-1} - 1)}{1 - (2 - \varepsilon)\theta}, \tag{28}$$

and odd otherwise.

Figure 1 illustrates the possibility of multiple steady states. The curves $LHS(z)$ and $RHS(z)$ represent the left-hand side and the right-hand side of (27), respectively. While $RHS(z)$ is a strictly increasing function of z, $LHS(z)$ is U-shaped and reaches the minimum at z_m. Also, $LHS(z_0) > RHS(z_0)$ at the lowest admissible value z_0. When $\gamma < \gamma_E$, $LHS(z_m) < RHS(z_m)$ and so there is at least one solution to (27). An even number of solutions exist if $LHS(z)$ exceeds $RHS(z)$ for large z. This is the case when the parameter values satisfy (28). Without loss of generality, I will assume that there are only two steady states when the number of steady states is even. The steady state with a lower value of z^* is labelled EH and the steady state with a higher value of z^* is labelled EL.

The aggregate level of real activities is higher in equilibrium EH than in equilibrium EL. In particular, the levels of consumption, output, capital, labor input and the real money balance are all higher in equilibrium EH than in equilibrium EL. For this reason, I call equilibrium EH the "high-activity" equilibrium and equilibrium EL the "low-activity" equilibrium. The main reason why equilibrium EH has higher aggregate activities than equilibrium EL is that the number of sellers is higher in equilibrium EH, which produces more desirable matches and trades in equilibrium EH than in equilibrium EL. For this reason, the high-activity equilibrium has a high liquidity level than the low-activity equilibrium.[13]

Why are there multiple steady states? The answer lies in the endogenous division of the match surplus between buyers and sellers. The expectations of the degree to which the trading restrictions on money bind

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[13]One way to make this association between liquidity and aggregate activities more precise is to use the velocity of money to measure liquidity. Let p be the nominal price level. Then $p = m/(bq)$. The output velocity of money is $p\alpha bnq/m = \alpha n$, which increases with the number of sellers in the economy.

FIG. 1. Multiple monetary steady states and their responses to an increase in money growth
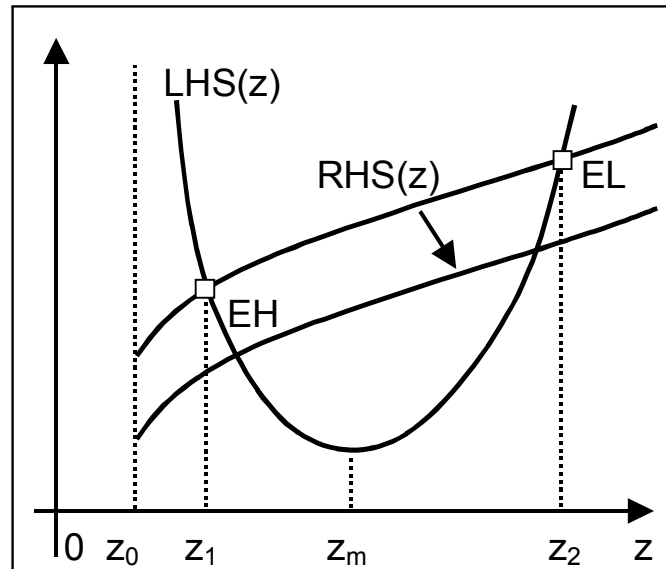
affect households' choices of the number of sellers, which in turn affect the degree to which the trading restrictions bind. Suppose, for example, that households expect that trades will be less constrained by money holdings. In this case, sellers' share of the match surplus will increase (see (16)). Anticipating this higher share for sellers, households increase the fraction of sellers. At the aggregate level, this increases the supply of goods and increases the purchasing power of money. A higher purchasing power of money implies that the trading restrictions on money will be less binding, which fulfills the initial expectations. Similarly, if households expect that the trading restrictions on money will be more binding, then sellers' share of the match surplus will be low and households will reduce the fraction of sellers, which will reduce aggregate supply of goods, reduce the purchasing power of money and indeed make the trading restrictions on money more binding. Therefore, the liquidity level in the economy depends critically on expectations as well as on the fundamentals of the economy.

The condition for multiple monetary equilibria, (28), is not a strong requirement. Under Assumption 1 on θ, for example, the condition is always satisfied for $u(c) = u_0 c^\delta$, where $\delta \in (0, 1]$, which has often been used in previous search models of money. Notice that this includes the case

where the marginal utility of consumption is constant. Also, if $\lim_{c \to 0} cu'(c)$ is finite, the condition is always satisfied when θ is sufficiently close to the upper bound $1/(2-\varepsilon)$. Also, since consumption and leisure are separable in the utility function (1), multiplicity here does not require that consumption and leisure be substitutable or complementary.[14]

Suppose that $\lim_{c \to 0} cu'(c)$ is positive and finite. Then, multiple monetary equilibria are more likely to exist for large θ than for small θ, and more likely for large γ than for small γ. When θ is close to the upper bound $1/(2 - \varepsilon)$, for example, the fraction of sellers is close to 0. In this case, households' choices of the fraction of sellers are very sensitive to changes in the beliefs of how binding the trading restrictions on money are. A small change in such beliefs induces households to change the fraction of sellers significantly which fulfills the change in beliefs. A similar explanation applies to the role of γ for multiplicity. If γ is sufficiently larger than β, for example, households are severely constrained by the trading restrictions on money, and so changes in the beliefs about how severely these restrictions bind lead to large changes in the number of matches that support the beliefs.

Multiple monetary steady states arise in my model from a distinct non-Walrasian feature – buyers' and sellers' shares of the match surplus in bilateral bargaining. This source of multiplicity is novel in comparison with Walrasian monetary models.[15] Previous search models of money also generated multiple equilibria (e.g., Shi, 1995, and Trejos and Wright, 1995), but multiplicity there depends critically on the existence of barter and indivisible money. The current model establishes multiple monetary equilibria without these elements, in particular, without the artificial assumption of indivisible money. Nevertheless, there is a similarity between the multiple equilibria here and those in previous search models. In Shi (1995), for example, beliefs about the value of the match surplus affect the surplus division which in turn supports the beliefs. When the total surplus from exchange is expected to be low, for example, buyers extract the entire match surplus, which makes sellers reduce the quantity of goods produced in each match and indeed leads to a low match surplus in equilibrium. It is interesting to note that, in both types of models, the size of the match

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[14]It is also apparent that multiplicity here relies on n being endogenous and θ to be in the interior of $(0, 1)$. In a similar framework, I showed that the monetary steady state is unique if n and b are both exogenous (Shi, 1997), or if $\theta = 1$ and only b is endogenous (Shi, 1999).

[15]A monetary economy with Walrasian markets can oscillate between different states even when there is no intrinsic shock to preferences or technology (see Grandmont, 1985, Woodford, 1986, and Matsuyama, 1990). In contrast to multiple steady states established here, such models often have a unique monetary steady state and the oscillations are due to either complicated dynamics (e.g., limited cycles) or extrinsic shocks (e.g., sunspots).

surplus and buyers' share of the surplus are inversely related to each other. Thus, buyers can get a larger *size* of surplus in the equilibrium where they extract a smaller *share* of the match surplus.

5.2. Effects of Money Growth

When there are an even number of steady states, an increase in the money growth rate (γ) has different effects on different steady states. The following corollary summarizes these effects (see Appendix 1 for a proof):[16]

COROLLARY 1. *In the high-activity steady state (the one with a low z^*),*

$$\frac{dz^*}{d\gamma} > 0, \ \frac{dq^*}{d\gamma} < 0, \ \frac{d(k^*/n^*)}{d\gamma} < 0, \ and \ \frac{dl^*}{d\gamma} < 0.$$

In the low-activity steady state, the effects of γ on $(z^*, q^*, k^*/n^*, l^*)$ are opposite to the above. In both steady states, n^*, c^* and aggregate output increase with γ. In the low-activity steady state, $dk^*/d\gamma > 0$. In the high-activity steady state, $dk^*/d\gamma > 0$ only if γ is not too close to γ_E.

An increase in the money growth has opposite intensive effects in the two steady states, i.e., the quantities of trade in each match respond to money growth in opposite directions in the two steady states. To see this, note that an increase in γ shifts down the curve $RHS(z)$ (see Figure 1) and so changes the value of z^* in the two steady states in opposite directions. In the high-activity equilibrium EH, the increase in γ reduces each buyer's real money balance q^* ($= bq^*$). Each seller's capital k^*/n^* and labor input l^* also fall. In contrast, in the low-activity equilibrium EL, the three variables $(q^*, k^*/n^*, l^*)$ all increase in response to an increase in γ.

These differences in the intensive effects are intuitive. In the low-activity equilibrium, agents are severely liquidity constrained. An increase in the money growth rate provides additional liquidity and so the trading restrictions on money become less binding, because $z'(\gamma) < 0$ in such an equilibrium. As the cost of the money restrictions falls, sellers' share of the match surplus increases and so sellers increase the labor input and produce more goods for money in each match. In the high-activity equilibrium, however, agents are not very severely constrained by liquidity. In this case, the inflation consequence of a high money growth rate dominates and so each seller reduces the quantity of goods produced for money in each match.

Despite the differences in the intensive effects, money growth has the same positive extensive effects in the two steady states. An increase in the money growth increases the number of sellers in both steady states

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[16]When there are an odd number of steady states, the one with the lowest z^* behaves in the same way as the high-activity steady state described in the corollary.

and hence increases the number of desirable matches. In the low-activity equilibrium, this positive extensive effect reinforces the positive intensive effects of money growth, and so aggregate consumption and output increase with the money growth rate. In the high-activity equilibrium, the positive extensive effect dominates the negative intensive effects of money growth, again leading to higher aggregate consumption and output. Therefore, this model can produce a positive relationship between money growth and aggregate output in both steady states, and the key to this positive relationship is the positive liquidity effect of money growth.

To explain why an increase in the money growth increases the number of sellers in both steady states, let me start with households' optimal choice of the money balance, characterized by (20). This condition requires that the capital loss to holding money be offset by money's role of relaxing the trading restrictions. In steady states, the capital loss to holding money is $(\gamma\beta^{-1} - 1)$. The role of money in exchange is quantified by $\theta\alpha n^* z^*$, where $\theta\alpha n^*$ is the probability with which a buyer faces the trading restriction on money. An increase in the money growth rate increases the capital loss to holding money. To compensate for this loss, money must have an increased role in relaxing the trading restrictions. If such restrictions become less binding as a result of an increase in the money growth rate (i.e., if z^* falls), as in the low-activity equilibrium, then agents must face the trading restrictions more frequently than before. This requires n^* to increase. If the trading restrictions become more binding (i.e., if z^* increases), then the response of n^* depends in general on the magnitude in which z^* increases. Because the trading restrictions on money are not very binding in the high-activity equilibrium, an increase in the money growth rate does not increase z^* much. To compensate for the higher capital loss to money, the number of transactions must increase.

The above comparison between the two steady states illustrates two important features of this search model. First, the positive extensive effects of money growth are robust across different equilibria, in contrast to the intensive effects. Second, an increase in the money growth rate has stronger effects on real aggregate output in the low-activity equilibrium than in the high-activity equilibrium, because the extensive and intensive effects work in the same direction in the low-activity equilibrium but opposite directions in the high-activity equilibrium. Because liquidity is lower in the low-activity equilibrium than in the high-activity equilibrium, this result suggests that expansionary monetary policies are more useful in stimulating output when liquidity is low than when liquidity is high.

Finally, the welfare effect of money growth is ambiguous analytically. While a higher money growth rate increases consumption, it also increases the cost of participating in exchange by increasing the number of sellers. When the utility function of consumption exhibits constant relative risk

aversion, numerical examples seem to suggest that a higher money growth rate increases steady state welfare when the money growth rate is small.

6. CONCLUSION

In this paper I construct a search monetary model with capital accumulation where money and goods are both divisible. Agents in matches determine the terms of trade through a sequential bargaining process and they face trading restrictions that require the quantity of money traded not to exceed what the buyer brings into the match. I show that sellers' share of the match surplus decreases with the severity of the trading restrictions. Such endogenous surplus shares generate multiple, self-fulfilling monetary steady states. When agents expect that the trading restrictions will not bind severely, sellers' surplus share is large and so households allocate more members to be sellers. This leads to a larger number of desirable matches, higher aggregate supply of goods, and higher purchasing power of money, which fulfills the expectations of not severely binding trading restrictions. On the other hand, when agents expect that the trading restrictions will bind severely, they reduce the number of sellers, which leads to lower aggregate output and lower purchasing power of money, again fulfilling the expectations. In both steady states, an increase in the money growth rate increases aggregate output and consumption by increasing the number of matches.

The current model interprets liquidity as the number of transactions. This interpretation provides a useful link between aggregate activities and liquidity. Aggregate activities are higher in the equilibrium with high liquidity than in the equilibrium with low liquidity. Also, an expansionary monetary policy increases aggregate output in both equilibria because the policy increases the liquidity level in both equilibria. Moreover, the model suggests that such positive effects of money growth on real output are stronger in an economy where liquidity is severely constrained than in an economy where liquidity is not constrained.

To test the model, one can examine cross-country evidence to see whether expansionary monetary policies are more effective in low-income countries than in high-income countries, after controlling for other differences between countries. Another way to test the model is to interpret the two steady states as states of a stochastic economy where there are intrinsic or extrinsic shocks. With this interpretation, the low-activity state is a recession and the high-activity state is a boom. Then, the current model implies that sellers' share of the match surplus is procyclical. To test this implication, one can examine whether firms' mark-ups are procyclical.

APPENDIX

Proof of Proposition 1

By the definition of a symmetric monetary equilibrium, $\omega_t m_t > 0$ for all t, which implies that the value function $v(k_{t+1}, m_{t+1})$ is increasing in m_{t+1}. Thus the constraints (15), (8) and (12) all hold with equality. Use these equalities to eliminate m_{t+1}, x_t^b and x_t^s in the household's decision problem (PH). Deriving the following first-order conditions for q_t^b and q_t^s, and replacing capital-case variables with lower-case letter under symmetry, I obtain the following:

$$u'(c_t) = \frac{\sigma}{\varepsilon} \cdot \frac{\omega_t + \lambda_t}{\omega_t} \cdot \frac{\Phi\left(q_t^b\right)}{q_t^b}, \tag{A.1}$$

$$u'(c_t) = \frac{\sigma}{\varepsilon} \cdot \frac{\omega_t}{\omega_t - \pi_t} \cdot \frac{\Phi\left(q_t^s\right)}{q_t^s}, \tag{A.2}$$

where $\Phi(q_t) = \Phi\left(\frac{k_t}{n_t} f\left(\frac{n_t q_t}{k_t}\right)\right)$.

Next, substitute (10) and (11) into the equality forms of (8) and (12). Imposing symmetry, I have:

$$\omega_t x_t^b - \Phi\left(q_t^b\right) = \frac{(1-\theta)\beta^\Delta}{1 - \theta\beta^\Delta}\left[\omega_t x_t^s - \Phi\left(q_t^s\right)\right], \tag{A.3}$$

$$u'(c_t)q_t^s - \omega_t x_t^s = \frac{\theta\beta^\Delta}{1 - (1-\theta)\beta^\Delta}\left[u'(c_t)q_t^b - \omega_t x_t^b\right]. \tag{A.4}$$

The match surplus $u'(c) - \Phi(q)$ is positive in a symmetric monetary equilibrium (otherwise money would not be valued), and so there is an immediate agreement in sequential bargaining.

I now show that λ and π are either both positive or both zero in symmetric monetary equilibria, i.e., it is never the case that $\lambda > 0$ and $\pi = 0$ or that $\lambda = 0$ and $\pi > 0$. To see this, suppose, to the contrary, that $\lambda_t > 0$ and $\pi_t = 0$. In this case, (A.1) and (A.2) imply

$$\frac{\Phi(q_t^b)/q_t^b}{\Phi(q_t^s)/q_t^s} = \frac{\omega_t}{\omega_t + \lambda_t} < 1.$$

Since $\Phi(q)/q$ is an increasing function of q, the above relation implies $q_t^b < q_t^s$. Notice that the strict inequality holds no matter how small Δ is. Since $\lambda_t > 0$, $x_t^b = m_t/b$. Taking the limit $\Delta \to 0$ on (A.3), I have

$$\omega_t\left(\frac{m_t}{b} - x_t^s\right) \to \Phi(q_t^b) - \Phi(q_t^s) < 0,$$

which contradicts the trading restriction $x_t^s \leq m_t/b$. Thus, the combination $\lambda_t > 0$ and $\pi_t = 0$ is inconsistent with a symmetric monetary equilibrium. Similarly, the combination $\lambda_t = 0$ and $\pi_t > 0$ is inconsistent with a symmetric monetary equilibrium.

To establish the rest of the proposition, suppose first that $\lambda_t > 0$ and $\pi_t > 0$. In this case, $x_t^s = x_t^b = m_t/b$ and (A.3) implies that $q_t^s = q_t^b$ when $\Delta \to 0$. With $q_t^s = q_t^b = q_t$, (A.1) and (A.2) imply (18). It remains to show that D_t^s and D_t^b are given by (16) and (17) in the limit $\Delta \to 0$. To achieve this, solve q_t^s from (A.4):

$$q_t^s = J(q_t^b, \Delta) \equiv \frac{\theta \beta^\Delta}{1 - (1-\theta)\beta^\Delta} q_t^b + \left[1 - \frac{\theta \beta^\Delta}{1 - (1-\theta)\beta^\Delta}\right] \frac{\omega_t x_t}{u'(c_t)}.$$

Clearly, $J(q_t^b, 0) = q_t^b$. Substituting q_t^s into (A.3) yields

$$\omega_t x_t = \left[\Phi(q_t^b) - \frac{(1-\theta)\beta^\Delta}{1 - \theta\beta^\Delta} \Phi(J(q_t^b, \Delta))\right] \bigg/ \left[1 - \frac{(1-\theta)\beta^\Delta}{1 - \theta\beta^\Delta}\right].$$

When Δ approaches zero, both the numerator and the denominator on the right-hand side of the above equality approach zero. Applying l'Hopital's rule and using (A.1) yields (16). Substituting $\omega_t x_t$ from (17) into (11) and taking the limit $\Delta \to 0$ yields (17).

The other possible case $\lambda_t = \pi_t = 0$ can be examined similarly and the same equations (16) – (18) hold, with $\lambda_t = \pi_t = 0$. This completes the proof for Proposition 1. ∎

Proofs of Proposition 2 and Corollary 1

For Proposition 2, I show first that z^* is determined by (27). With the notation $z = \lambda/\omega$ and the function $R(z)$ defined by (24), (21) implies $\Phi^* = R(z^*)$ and (18) implies

$$u'(c^*) = \frac{\sigma}{\varepsilon}(1 + z^*) \frac{R(z^*)}{q^*}.$$

Substituting u' from the above into the steady state version of (19) yields

$$\frac{n^* q^*}{k^*} = \frac{(\beta^{-1} - 1)(1 + z^*)}{\alpha b(1 - \varepsilon)}.$$

This equation and $\Phi^* = R(z^*)$ jointly yield $k^*/n^* = K(z^*)$ and $q^* = q(z^*)$, where $K(\cdot)$ is defined by (25) and $q(\cdot)$ by (26). Substituting $c^* = \alpha b n^* q^*$ from (14) into (19) yields:

$$\frac{R(z^*)}{K(z^*)} = \frac{\varepsilon(\beta^{-1} - 1)}{\alpha b \sigma(1 - \varepsilon)} u'\left(\alpha b n^* q(z^*)\right).$$

To obtain (27) from the above equation, it suffices to show the following:

$$n^* = (\gamma/\beta - 1)/z^*.$$

Using (16), I get $x_{t+1}/x_t = \omega_t/\omega_{t+1}$ in the steady state. Since $x_{t+1}/x_t = m_{t+1}/m_t = \gamma$, then $\omega_t/\omega_{t+1} = \gamma$, and so (20) yields the desired relation $n^* = (\gamma/\beta - 1)/z^*$.

Now I examine the solutions to (27). It is easy to verify that $K'(z) < 0$, $R'(z) < 0$ and $q'(z) < 0$. Thus, the right-hand side of (27), denoted $RHS(z)$, is an increasing function of z and a decreasing function of γ. Also, $RHS(z) \to 0$ when $z \to z_0$, where z_0 is defined in (22). The left-hand side of (27), denoted $LHS(z)$, does not depend on γ once z is given. $LHS'(z) > 0$ if and only if

$$g(z) \equiv -\frac{(1+z)R'(z)}{R(z)} < \frac{\sigma}{\varepsilon(\sigma - 1)}. \tag{A.5}$$

It can be verified that the function $g(z)$ defined above is monotonically decreasing in z for all $z > z_0$, with $g(z_0) = \infty$ and $g(\infty) = 0$. Thus there exists a unique level z_m, with $z_m > z_0$, such that $LHS'(z) > 0$ if and only if $z > z_m$.

Since $LHS(z_0) = \infty > RHS(z_0)$ and $LHS(z)$ reaches the minimum at $z = z_m$, (27) has at least one solution if $LHS(z_m) < RHS(z_m)$ (see Figure 1), which can be rewritten as:

$$\gamma < \gamma_{E0} \equiv \beta \left[1 + \frac{z_m}{\alpha b q(z_m)} u'^{-1} \left(\frac{\alpha b \sigma (1 - \varepsilon) R(z_m)}{\varepsilon(\beta^{-1} - 1)K(z_m)} \right) \right].$$

Since $LHS(z)$ is U-shaped and $RHS(z)$ is monotonically increasing, the number of solutions to (27) is even if and only if $\lim_{z \to \infty} LHS(z)/RHS(z) > 1$, which is equivalent to (28). When $\gamma < \gamma_{E0}$, the smallest solution to (27) is less than z_m and hence behaves in the same way regardless of whether (27) has an even or odd number of solutions. Denote the largest solution for z^* by $z_2(\gamma)$ and the least solution by $z_1(\gamma)$. Then, $z_0 < z_1(\gamma) < z_m < z_2(\gamma)$.

There are other restrictions on γ. Since $n^* = (\gamma/\beta - 1)/z^*$ and $z^* > 0$, then $n^* > 0$ requires $\gamma > \beta$ and $n^* < 1 - b$ requires $z^* > (\gamma/\beta - 1)/(1 - b)$. Consider the requirement $z_1(\gamma) > (\gamma/\beta - 1)/(1 - b)$. It is satisfied for γ close to β since $z_1(\beta) = z_0 > 0$. Suppose there is a level γ_{E1} such that the inequality becomes an equality (if such γ does not exist, let $\gamma_{E1} = \gamma_{E0}$). Then the steady state $z_1(\gamma)$ satisfies $n^* < 1 - b$ for all $\gamma \in (\beta, \min(\gamma_{E0}, \gamma_{E1}))$. Similarly one can define γ_{E2} and show that the steady state $z_2(\gamma)$ satisfies $n^* < 1 - b$ for all $\gamma \in (\beta, \min(\gamma_{E0}, \gamma_{E2}))$. Thus, all solutions to (27) satisfy the equilibrium requirements if $\beta < \gamma < \gamma_E \equiv \min(\gamma_{E0}, \gamma_{E1}, \gamma_{E2})$. This completes the proof of Proposition 2.

The proof of Corollary 1 is as follows. Since $LHS(z)$ does not depend on γ directly, then

$$\frac{dz^*}{d\gamma} = \frac{\partial RHS(z)/\partial \gamma}{LHS'(z) - RHS'(z)}.$$

Because $\partial RHS(z)/\partial \gamma < 0$, $dz^*/d\gamma > 0$ iff $LHS'(z) < RHS'(z)$. Under the restriction $\gamma < \gamma_E$, z_1 and z_2 lie on opposite sides of z_m and so $LHS'(z) < RHS'(z)$ iff $z < z_m$. Thus, $z_1'(\gamma) > 0$ and $z_2'(\gamma) < 0$. Since $q'(z) < 0$, $K'(z) < 0$ and $R'(z) < 0$, q^*, k^*/n^* and l^* are all decreasing functions of γ in the steady state with $z_1(\gamma)$ but are increasing functions of γ in the steady state with $z_2(\gamma)$. For the response of c^*, note that the expression in $u'(.)$ in $RHS(z)$ is c^*. Because $u'(.)$ is a decreasing function, $dc^*/d\gamma > 0$ iff $dRHS(z)/d\gamma < 0$. Compute

$$\frac{dRHS(z)}{d\gamma} = \frac{\partial RHS(z)}{\partial \gamma} + RHS'(z)\frac{dz^*}{d\gamma} = \frac{LHS'(z) \cdot \partial RHS(z)/\partial \gamma}{LHS'(z) - RHS'(z)}.$$

Recall that $\partial RHS(z)/\partial \gamma < 0$. In the equilibrium with $z^* = z_1(\gamma)$, $LHS'(z) < 0 < RHS'(z)$ and so $dRHS(z)/d\gamma < 0$; in the equilibrium with $z^* = z_2(\gamma)$, $LHS'(z) > RHS'(z) > 0$ and so again $dRHS(z)/d\gamma < 0$. Thus, $dc^*/d\gamma > 0$ in both steady states. In steady states, aggregate output is equal to consumption because capital does not depreciate. Therefore, aggregate output increases with γ in both steady states.

For the response of n^*, recall that $n^* = (\gamma/\beta - 1)/z^*$. In the steady state with $z_2(\gamma)$, $dz^*/d\gamma < 0$ implies $dn^*/d\gamma > 0$. For the steady state with $z_1(\gamma)$, recall that $c^* = \alpha b n^* q^*$ and so $n^* = c^*/(\alpha b q^*)$. Then $dq^*/d\gamma < 0$ and $dc^*/d\gamma > 0$ imply $dn^*/d\gamma > 0$. For the response of k^*, direct calculation reveals:

$$\frac{dk^*}{d\gamma} = \frac{1}{1+z^*} \cdot \left(\frac{dz^*}{d\gamma}\right) \cdot \left[\left(\frac{\sigma - 1}{\sigma}g(z^*) - \frac{1}{\varepsilon}\right)\rho - 1\right],$$

where $\rho \equiv -u'/(c^*u'')$. In the equilibrium with $z_2(\gamma)$, $z^* > z_m$ implies $g(z^*) < \sigma/[\varepsilon(\sigma - 1)] = g(z_m)$ and so $dk^*/d\gamma > 0$. In the equilibrium with $z_1(\gamma)$, $dk^*/d\gamma > 0$ if and only if $g(z^*) > g(z_m)(1 + \varepsilon/\rho)$, i.e., if and only if γ is not too close to γ_{E0}. ∎

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