ANNALS OF ECONOMICS AND FINANCE **1**, 359–381 (2000)

Optimal Portfolios in an Incomplete Market *

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A self-financing optimal investment problem is considered in an incomplete market. The general existence of optimal portfolios is discussed via variational method of stochastic optimal control and the theory of (forward-) backward stochastic differential equations.

Key Words: :Optimal portfolio; Stochastic control; Backward stochastic differential equations.

JEL Classification Numbers: G0, G1, G6.

1. 1. FORMULATION OF THE PROBLEM

We consider the Black-Scholes market model (see Karatzas and Shreve (1998) and Musiela and Rutkowski (1997)): There are $(n+1)$ assets continuously traded in the market. The 0-th asset is a bond, and the last n are stocks. The price process of the i-th asset is denoted by $P_i(\cdot)$ and the following system of stochastic differential equations (SDEs, for short) is satisfied by $P_i(\cdot)$'s:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, \\ dP_i(t) = b_i(t)P_i(t)dt + P_i(t)\displaystyle\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t), \qquad 1 \le i \le n, \\ P_i(0) = p_i, \qquad 0 \le i \le n, \end{cases} \tag{1.1}$$

where $r(\cdot)$, $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are called the *interest rate* (of the bond), the *appreciation rate*, and the *volatility* (of the stocks), respectively, $W(\cdot) \equiv$

* This work is supported in part by the NSFC, under grant 79790130, the National Distinguished Youth Science Foundation of China under grant 19725106, and the Chinese Education Ministry Science Foundation under grant 97024607.

1529-7373/2000

$(W_1(\cdot), \cdots, W_d(\cdot))$ is a d-dimensional standard Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(\cdot)$. We denote $b(\cdot) = (b_1(\cdot), \cdots, b_n(\cdot))^T$ and $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times d}$.

According to Musiela and Rutkowski (1997), an equivalent condition for the completeness of the market described above is $n \geq d$ and $\sigma(t)$ being of full rank for all $t \in [0, T]$, almost surely. In the current paper, we do not assume the full rank condition for $\sigma(t)$. As a matter of fact, we even assume neither $n \geq d$ nor $n < d$. Thus, the market under our consideration is *incomplete* in general.

Next, we introduce the so-called *log-price* process $X_i(t) = \ln P_i(t)$ for the i-th asset, and using Itô's formula, we have (note (1.1))

$$\begin{cases} dX_0(t) = r(t)dt, \\ dX(t) = \big[b(t) - \hat{\sigma}(t)\big]dt + \sigma(t)dW(t), \\ X_0(0) = x_0 \triangleq \ln p_0, \ \ X(0) = x_0 \triangleq (\ln p_1, \cdots, \ln p_n)^T, \end{cases} \tag{1.2}$$

where

$$\hat{\sigma}(\cdot) \equiv (\hat{\sigma}_1(\cdot), \cdots, \hat{\sigma}_n(\cdot)), \quad \hat{\sigma}_i(t) = \frac{1}{2}\sum_{j=1}^{d} |\sigma_{ij}(t)|^2, \quad 1 \leq i \leq n.$$

Motivated by the above, in what follows, we consider the following market model which is a little more general than (1.2).

$$\begin{cases} dX_0(t) = r(t, X(t))dt, \\ dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X_0(0) = x_0, \ \ X(0) = x. \end{cases} \tag{1.3}$$

One can similarly consider the case that $r(\cdot), b(\cdot), \sigma(\cdot)$ also depend on $X_0(\cdot)$, which does not bring any additional difficulty to the problems under our consideration.

Now, let us consider an investor who has an initial wealth $y \in \mathbb{R}$ (also called *initial endowment*). He invests this amount in the market described as (1.1). At any time $t \in [0, T]$, the total wealth is denoted by $Y(t)$. We call $Y(\cdot)$ the *wealth process* of this investor (and thus, $Y(0) = y$). Let us assume that the investment is *self-financing*, meaning that, besides the initial endowment y, there is no money brought in or taken out during the time interval $[0, T]$. At any time $t \in [0, T]$, the total wealth is decomposed into $(n + 1)$ parts:

$$Y(t) = \sum_{i=0}^{n} \pi_i(t), \qquad t \in [0, T], \tag{1.4}$$

where $\pi_i(t)$ is the market value of the i-th asset held by the investor, $0 \le i \le n$ with $\pi_0(t) \triangleq Y(t) - \sum_{i=1}^n \pi_i(t)$. We call $\pi(\cdot) \triangleq (\pi_1(\cdot), \cdots, \pi_n(\cdot))$ a *portfolio process*, which determines the investment manner of the investor. A standard computation shows that $Y(\cdot)$ satisfies the following equation: (Karatzas and Shreve (1998) and Musiela and Rutkowski (1997))

$$\begin{cases} dY(t) = \big\{r(t,X(t))Y(t) + \langle h(t,X(t)),\pi(t)\rangle\big\}dt \\ \qquad\qquad +\langle \pi(t), \sigma(t,X(t))dW(t)\rangle, \quad t \in [0,T], \\ Y(0) = y, \end{cases} \tag{1.5}$$

where

$$h(t,x) \triangleq b(t,x) - r(t,x)\mathbf{1}, \qquad (t,x) \in [0,T] \times \mathbb{R}^n, a.s., \tag{1.6}$$

with $\mathbf{1} \triangleq (\mathbf{1},\cdots,\mathbf{1})^{\mathbf{T}} \in \mathbb{R}^{\mathbf{n}}$. We will see that some of the discussions below remain true for (1.5) without assuming relation (1.6). However, for definiteness, we keep relation (1.6) below. For convenience, we refer to the triple (r,b,σ) appeared in (1.3) as a market.

Before going further, let us introduce the following basic assumption:

(A1) Random fields $r : [0,T] \times \mathbb{R}^n \times \Omega \to [0,\infty)$, $b : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^{n\times d}$ satisfy the following: For any $x \in \mathbb{R}^n$, $t \mapsto (r(t,x), b(t,x), \sigma(t,x))$ is $\{\mathcal{F}_t\}_{t\ge 0}$-adapted and there exists a constant $L > 0$, such that for all $t \in [0,T]$, $x, \widetilde{x} \in \mathbb{R}^n$,

$$\begin{cases} |b(t,x) - b(t,\widetilde{x})| + |\sigma(t,x) - \sigma(t,\widetilde{x})| \le L|x - \widetilde{x}|, \quad a.s., \\ |r(t,x)| + |b(t,0)| + |\sigma(t,x)| \le L, \quad a.s. \end{cases} \tag{1.7}$$

Sometime, we need the following stronger assumption.

(A1)$'$ Maps $r : [0,T] \times \mathbb{R}^n \to [0,\infty)$, $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are smooth with bounded derivatives such that (A1) is satisfied.

We note that in (A1)$'$, functions r, b, σ are all deterministic. Clearly, (A1)$'$ is a special case of (A1). But still, (A1)$'$ is very general. Also, we will see that functions r, b and σ only need to be C^4. Here, we assume the functions to be smooth just for simplicity.

Under (A1), by a standard result for SDEs, we know that for any $x \in \mathbb{R}^n$, there exists a unique strong solution $X(\cdot)$ to (1.3). Hereafter, (A1) will be assumed, and thus $X(\cdot)$ is uniquely determined (once $x \in \mathbb{R}^n$ is given). Next, we introduce the following set:

$$\Pi[0,T] \triangleq \Big\{ \begin{array}{l} \pi : [0,T] \times \Omega \to \mathbb{R}^n \;\big|\; \pi(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\ge 0}\text{-adapted}, \\ \langle h(\cdot\,, X(\cdot)), \pi(\cdot)\rangle,\ |\sigma(\cdot\,, X(\cdot))^T \pi(\cdot)|^2 \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \end{array} \Big\}, \tag{1.8}$$

where $L^1_{\mathcal{F}}(0,T;\mathbb{R})$ is the set of all $\{\mathcal{F}_t\}_{t\ge 0}$-adapted processes $\varphi(\cdot)$ such that $E\int_0^T |\varphi(s)|ds < \infty$. It is clear that the set $\Pi[0,T]$ depends on the random fields r, b, σ as well as the initial log-price x. Any $\pi(\cdot) \in \Pi[0,T]$ is called a *feasible portfolio*. We know that under (A1), for any $y \in \mathbb{R}$ and $\pi(\cdot) \in \Pi[0,T]$, (1.5) admits a unique strong solution $Y(\cdot) \equiv Y(\cdot\,; y, \pi(\cdot))$ which admits the following representation:

$$\begin{aligned} Y(t) = &\; e^{\int_0^t r(\tau,X(\tau))d\tau} y + \int_0^t e^{\int_s^t r(\tau,X(\tau))d\tau} \langle h(s,X(s)), \pi(s)\rangle ds \\ &+ e^{\int_0^t r(\tau,X(\tau))d\tau} \int_0^t e^{-\int_0^s r(\tau,X(\tau))d\tau} \langle \pi(s), \sigma(s,X(s)) dW(s)\rangle, \quad t \in [0,T]. \end{aligned} \tag{1.9}$$

Now, for this particular investor, he has his own attitude to the risk versus the gain at the final time T, which can be described by a strictly increasing and concave utility function $g : \mathbb{R} \to [-\infty, \infty)$. The investor would like to maximize the following expected payoff:

$$J(y; \pi(\cdot)) = E\big[g(Y(T; y, \pi(\cdot)))\big] \tag{1.10}$$

by choosing a suitable portfolio $\pi(\cdot) \in \Pi[0,T]$. To be more precise, let us state the following problem.

Problem (C). For given initial endowment $y \in \mathbb{R}$, find a portfolio $\overline{\pi}(\cdot) \in \Pi[0,T]$ such that

$$J(y; \overline{\pi}(\cdot)) = \sup_{\pi(\cdot)\in\Pi[0,T]} J(y; \pi(\cdot)). \tag{1.11}$$

Any $\overline{\pi}(\cdot) \in \Pi[0,T]$ satisfying (1.11) is called an *optimal portfolio*, the corresponding $\overline{Y}(\cdot) \triangleq Y(\cdot\,; y, \overline{\pi}(\cdot))$ is called an *optimal wealth process*, and $(\overline{Y}(\cdot), \overline{\pi}(\cdot))$ is called an *optimal pair*. Problem (C) can be regarded as a *stochastic optimal control problem* (Musiela and Rutkowski (1997)). In that context, people refer to (1.5) as the *state equation*, $Y(\cdot)$ as the *state process* and $\pi(\cdot)$ as the *control*.

In general, the range of the function $g(\cdot)$ contains the whole $(-\infty, \infty)$ and the wealth process $Y(\cdot\,; y, \pi(\cdot))$ is linear in the portfolio $\pi(\cdot)$ whose values runs over the whole $\mathbb{R}^n$. Thus, a direct approach using minimizing sequence to prove the existence of an optimal portfolio is not applicable for our problem. Moreover, we even allow $g(\cdot)$ to take $-\infty$ as its values at some points in $\mathbb{R}$. The above difficulties make the existence of optimal portfolios of Problem (C) non-trivial. Our approach is as follows: We first derive a set of necessary conditions for optimal portfolios. Then, by the concavity of $g(\cdot)$ and the linearity of the state equation (1.5), we show that the set of the obtained necessary conditions is also sufficient for optimality. Next, by a comparison theorem for backward stochastic differential equations, we show the existence of the portfolio satisfying the necessary conditions, which leads to the existence of an optimal portfolio. This also gives a construction of an optimal portfolio. Finally, for a special form of utility function, we construct an optimal portfolio via a Riccati type equation which is a BSDE with a quadratic adjustment term.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminaries. Necessary and sufficient optimality conditions, as well as the existence of optimal portfolios are established in Section 3. Section 4 presents a construction of an optimal portfolio via a Riccati type equation.

The readers are referred to Karatzas and Shreve (1998), Musiela and Rutkowski (1997), and Pliska (1997) and the extended references cited therein for some standard results about optimal investment problems. Also, the readers can find a different approach for the similar problem in Kramakov and Schachermayer (1999). For other related works, see El Karoui, Peng, and quenez (1997), Ma and Yong (1999), and Yong (1999, 2000), and Yong and Zou (1999).

2. 2. FEASIBILITY, FINITENESS AND SOLVABILITY

In this section, we present some preliminary results. First of all, let us introduce some notations. Let $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ be the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\int_0^T E|X(t)|^2 dt < \infty$, $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ be the set of all $\mathcal{F}_T$ measurable random variables ξ such that $E|\xi|^2 < \infty$. The definition of $L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is obvious. Now, let us say something about the function $g:\mathbb{R}\to[-\infty,\infty)$. We define the *domain* $\mathcal{D}(g)$ of $g(\cdot)$ as follows:

$$\mathcal{D}(g) \triangleq \{y \in \mathbb{R} \bigm| g(y) > -\infty\}, \tag{2.1}$$

and let

$$y_0 \triangleq \inf \mathcal{D}(g). \tag{2.2}$$

We call y_0 the *subsistence terminal wealth* (Karatzas and Shreve (1998)) for the investor. Next, we introduce the following assumption on $g(\cdot)$.

(A2) Function $g:\mathbb{R}\to[-\infty,\infty)$ is upper semicontinuous with the domain $\mathcal{D}(g)$ defined by (2.1), and (note (2.2))

$$(y_0,\infty) \subseteq \mathcal{D}(g) \subseteq [y_0,\infty). \tag{2.3}$$

Function $g(\cdot)$ is C^2 in (y_0,∞) with

$$g'(y) > 0, \quad g''(y) < 0, \quad \forall y \in (y_0,\infty), \tag{2.4}$$

$$\lim_{y\to\infty} g'(y) = 0, \quad \lim_{y\to y_0} g'(y) = \infty. \tag{2.5}$$

Moreover, the inverse $(g')^{-1}(\cdot):(0,\infty)\to(y_0,\infty)$ of $g'(\cdot)$ satisfies

$$|(g')^{-1}(z)| \leq C(|z|^\alpha + |z|^{-\alpha}), \quad \forall z \in (0,\infty). \tag{2.6}$$

Note that (2.4) implies that $g(\cdot)$ is strictly increasing and strictly concave; (2.3) means that the domain $\mathcal{D}(g)$ of $g(\cdot)$ is either (y_0,∞) or $[y_0,\infty)$ (if $y_0>-\infty$). It is not hard to check that any one of the following functions satisfies (A2) with $y_0=-\infty$, or 0.

$$\begin{cases} g(y)=1-e^{-\lambda y}, \quad y\in\mathbb{R}, \quad (\lambda>0), \\ g(y)=\begin{cases} y^\alpha, & y\ge 0, \\ -\infty, & y<0, \end{cases} \quad (0<\alpha<1), \\ g(y)=\begin{cases} \ln y, & y>0, \\ -\infty, & y\le 0. \end{cases} \end{cases} \tag{2.7}$$

We now introduce the following notion.

DEFINITION 2.1. (i) Problem (C) is said to be *feasible* at an initial endowment $y\in\mathbb{R}$ if there exists a $\pi(\cdot)\in\Pi[0,T]$ such that $J(y;\pi(\cdot))$ is a finite number. If the problem is feasible at any $y\in\mathbb{R}$, we say that Problem (C) is feasible.

(ii) Problem (C) is said to be *finite* at $y\in\mathbb{R}$ if

$$\sup_{\pi(\cdot)\in\Pi[0,T]} J(y;\pi(\cdot))\in(-\infty,\infty), \tag{2.8}$$

and, if the problem is finite at any $y\in\mathbb{R}$, we say that Problem (C) is finite.

(iii) Problem (C) is said to be (*uniquely*) *solvable* at $y\in\mathbb{R}$ if there exists a (unique) $\overline{\pi}(\cdot)\in\Pi[0,T]$ such that (2.8) holds, and if the problem is (uniquely) solvable at any $y\in\mathbb{R}$, we say that Problem (C) is (uniquely) solvable.

We denote $\mathcal{Y}_0$, $\mathcal{Y}_f$ and $\mathcal{Y}_s$ to be the sets of initial endowments at which Problem (C) is feasible, finite and solvable, respectively. It is clear that

$$\mathcal{Y}_s\subseteq\mathcal{Y}_f\subseteq\mathcal{Y}_0. \tag{2.9}$$

Let us present the following basic result.

PROPOSITION 2.1. *Let (A1)–(A2) hold.*

(i) *The set $\mathcal{Y}_0$ contains $\mathcal{D}(g)\bigcap[0,\infty)$.*

(ii) *If $\mathcal{Y}_f\ne\phi$, then*

$$\mathcal{Y}_f\supseteq(\inf\mathcal{Y}_f,\infty). \tag{2.10}$$

(iii) *If, in addition, function $g(\cdot)$ is bounded from above, then*

$$\mathcal{Y}_f=\mathcal{Y}_0. \tag{2.11}$$

(iv) *If the following holds:*

$$\lim_{y\to\infty} g(y) = +\infty, \tag{2.12}$$

and $\mathcal{Y}_f \neq \phi$, *then*

$$h(t,X(t)) \in \mathcal{R}\big(\sigma(t,X(t))\big) \triangleq \begin{array}{l}\{\sigma(t,X(t))\theta \mid \theta \in \mathbb{R}^d\},\\ a.e.t \in [0,T],\ a.s.\end{array} \tag{2.13}$$

(v) *If* $\mathcal{Y}_s \neq \phi$ *(without assuming (2.12)), then (2.13) holds.*

Proof. (i) For any $y \in \mathcal{D}(g)\bigcap[0,\infty)$, by taking $\pi(\cdot) = 0$, from (1.9), we see that

$$Y(T;y,0) \equiv e^{\int_0^T r(\tau,X(\tau))d\tau} y \geq y, \qquad a.s. \tag{2.14}$$

Thus, $Y(T;y,0) \in \mathcal{D}(g)$ which implies $y \in \mathcal{Y}_0$.

(ii) Let $y \in \mathcal{Y}_f$. For any $\hat{y} > y$, and $\pi(\cdot) \in \Pi[0,T]$, by the convexity of $g(\cdot)$, and the linearity of (1.9), we have

$$\begin{aligned} Eg(Y(T;\hat{y},\pi(\cdot))) &= Eg(e^{\int_0^T r(\tau,X(\tau))d\tau}(\hat{y}-y) + Y(T;y,\pi(\cdot))) \\ &\leq Eg(Y(T;y,\pi(\cdot))) + E\big[e^{\int_0^T r(\tau,X(\tau))d\tau}(\hat{y}-y)\big]. \end{aligned} \tag{2.15}$$

Thus, by the boundedness of $r(\cdot)$, and $y \in \mathcal{Y}_f$, we obtain

$$\begin{aligned} &\sup_{\pi(\cdot)\in\Pi[0,T]} Eg(Y(T;\hat{y},\pi(\cdot))) \\ &\leq \sup_{\pi(\cdot)\in\Pi[0,T]} Eg(Y(T;y,\pi(\cdot))) + e^{\|r\|_\infty T}(\hat{y}-y) < \infty. \end{aligned} \tag{2.16}$$

Since $y \in \mathcal{Y}_f$ is arbitrary, (2.10) follows.

(iii) It is obvious.

(iv) Since $\mathcal{Y}_f \neq \phi$, there exists some $y \in \mathcal{Y}_f$. Suppose (iv) does not hold. For any $t \in [0,T]$, let

$$\begin{aligned} G_t &\triangleq \{\omega \in \Omega \mid h(t,X(t)) \notin \mathcal{R}\big(\sigma(t,X(t))\big) = \mathcal{N}\big(\sigma(t,X(t))^T\big)^{\perp}\} \\ &= \{\omega \in \Omega \mid \exists \zeta \in \mathcal{N}\big(\sigma(t,X(t))^T\big), \langle h(t,X(t)),\zeta\rangle \neq 0\}. \end{aligned} \tag{2.17}$$

Then by a Filippov type theorem, we can find a $\zeta(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^n)$ such that

$$\begin{cases} \sigma(t,X(t))^T\zeta(t) = 0, \\ \big|\langle h(t,X(t)),\zeta(t)\rangle\big| \leq 1, \end{cases} \qquad a.e.t \in [0,T],\ a.s., \tag{2.18}$$

and

$$\mathbf{P}\Big(\big|\{t\in[0,T]\mid\langle h(t,X(t)),\zeta(t)\rangle\neq 0\}\big|>0\Big)=\int_0^T\mathbf{P}(G_t)dt>0,\quad(2.19)$$

with $|S|$ standing for the Lebesgue measure of $S\subseteq[0,T]$. Define

$$\pi(t)=\zeta(t)\mathrm{sgn}\big\{\langle h(t,X(t)),\zeta(t)\rangle\big\},\quad t\in[0,T],\qquad(2.20)$$

with $\mathrm{sgn}\,0\triangleq 0$. Then it follows from (2.18) that $\pi(\cdot)\in\Pi[0,T]$ and

$$\begin{cases}\sigma(t,X(t))^T\pi(t)=0,\quad a.e.t\in[0,T],\ a.s.,\\ \langle h(t,X(t)),\pi(t)\rangle=\big|\langle h(t,X(t)),\zeta(t)\rangle\big|.\end{cases}\qquad(2.21)$$

Now, we take $\pi^\lambda(\cdot)=\lambda\zeta(\cdot)\in\Pi[0,T]$, $\lambda>0$, and solve (1.5) to get

$$Y(T;y,\pi^\lambda(\cdot))=e^{\int_0^T r(\tau,X(\tau))d\tau}y+\lambda\int_0^T e^{\int_t^T r(\tau,X(\tau))d\tau}\big|\langle h(t,X(t),\zeta(t)\rangle\big|dt.\qquad(2.22)$$

Thus, by (2.12) and (2.19), we obtain

$$J(y;\pi^\lambda(\cdot))=E\big[g(Y(T;y,\pi^\lambda(\cdot)))\big]\to\infty,\quad(\lambda\to\infty),\qquad(2.23)$$

contradicting $y\in\mathcal{Y}_f$.

(v) Let $y\in\mathcal{Y}_s$, and let $\overline{\pi}(\cdot)\in\Pi[0,T]$ satisfy (1.11). Suppose again that (2.13) does not hold. Then we can find a $\zeta(\cdot)\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^n)$ satisfying (2.18) and (2.19). By taking $\pi(\cdot)\in\Pi[0,T]$ as (2.20), we have (2.21). Hence,

$$Y(T;y,\pi(\cdot)+\overline{\pi}(\cdot))=Y(T;y,\overline{\pi}(\cdot))+\int_0^T e^{\int_t^T r(\tau,X(\tau))d\tau}|\langle h(t,X(t)),\zeta(t)\rangle|dt.\qquad(2.24)$$

By (2.19), we get

$$J(y;\overline{\pi}(\cdot)+\pi(\cdot))>J(y;\overline{\pi}(\cdot)),\qquad(2.25)$$

contradicting the optimality of $\overline{\pi}(\cdot)$. ∎

The above result says that $\mathcal{Y}_0$ is always non-empty; condition (2.13) is necessary for $\mathcal{Y}_s$ to be non-empty; and if (2.12) holds, condition (2.13) is also necessary for $\mathcal{Y}_f$ to be non-empty. Moreover, in the case $\mathcal{Y}_f\neq\phi$, (2.10) holds. It is clear that (2.13) is equivalent to the existence of an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $\theta:[0,T]\times\Omega\to\mathbb{R}^n$ satisfying

$$h(t,X(t))=\sigma(t,X(t))\theta(t),\quad t\in[0,T],\ a.s.\qquad(2.26)$$

Let us now introduce the following assumption which is a little stronger than (2.26).

(A3) There exists a $\theta(\cdot) \in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^d)$ such that (2.26) holds.

Process $\theta(\cdot)$ in (2.26) is called a *risk premium*. When (A3) holds, (1.3) can be written as follows

$$\begin{cases} dY(t) = \big\{ r(t,X(t))Y(t) & +\langle \sigma(t,X(t))^T\pi(t), \theta(t)\rangle \big\} dt \\ & +\langle \sigma(t,X(t))^T\pi(t), dW(t)\rangle, \quad t \in [0,T], \\ \qquad Y(0) = y. \end{cases} \tag{2.27}$$

Condition (2.26) is closely related to an important notion about the market. For reader's convenience, let us briefly recall it.

Definition 2.2. A market (r,b,σ) is said to have *an arbitrage opportunity* on $[0,T]$ if there exists a $\pi(\cdot) \in \Pi[0,T]$ such that the solution $Y(\cdot\,;0,\pi(\cdot))$ of (1.5) (with $y=0$) satisfies

$$\begin{cases} Y(T;0,\pi(\cdot)) \geq 0, \quad a.s. \\ \mathbf{P}\big(Y(T;0,\pi(\cdot)) > 0\big) > 0. \end{cases} \tag{2.28}$$

If no arbitrage opportunity exists, we say that the market (r,b,σ) is of *no-arbitrage*.

The meaning of no-arbitrage is that there is no way of making riskless profit without initial investment. In the case that r, b, σ are all independent of x, one has another closely related notion — equivalent martingale measure. In the current case, provided (A1) holds, (1.3) admits a unique solution $X(\cdot)$, and we can regard $r(\cdot\,, X(\cdot))$, $b(\cdot\,, X(\cdot))$ and $\sigma(\cdot\,, X(\cdot))$ as known stochastic processes. Thus, the notion of equivalent martingale measure can be similarly introduced. More precisely, we have the following.

Definition 2.3. Let (A1) hold and $X(\cdot)$ be the solution of (1.3) (corresponding x). A probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ is called an *equivalent martingale measure* for the market (r,b,σ) if $\mathbf{Q}$ is equivalent to $\mathbf{P}$ and the price $\hat{P}_i(\cdot) \triangleq \frac{P_i(\cdot)}{P_0(\cdot)}$ of each *bond-discounted asset* is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$. Here, $P_i(t) = e^{X_i(t)}$, $0 \leq i \leq n$.

The following result links condition (2.26) with the notions in Definitions 2.2 and 2.3.

Proposition 2.2. *Let (A1). Then the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold for the following statements:*

(i) *There exists a* $\theta(\cdot)\in L^1_{\mathcal{F}}(0;\mathbb{R}^d)$ *satisfying*

$$E\Big\{e^{\frac{1}{2}\int_0^T|\theta(t)|^2dt}\Big\}<\infty, \tag{2.29}$$

such that (2.29) holds.

(ii) *There exists an equivalent martingale measure for the market.*

(iii) *The market is of no-arbitrage.*

(iv) *There exists an* $\{\mathcal{F}_t\}_{t\ge 0}$*-adapted process* $\theta:[0,T]\times\Omega\to\mathbb{R}^d$ *such that (2.26) holds.*

The proof of the above is very similar to the standard case (where r,b,σ are independent of $X(\cdot)$). We would like to point out that, in general, (i) and (iv) in the above are not necessarily equivalent. To see this, we need only take a $\beta(\cdot)\in L^1_{\mathcal{F}}(0,T;\mathbb{R})\setminus L^2_{\mathcal{F}}(0,T;\mathbb{R})$. Then

$$E\Big\{e^{\frac{1}{2}\int_0^T\beta(t)^2dt}\Big\}\ge\frac{1}{2}E\int_0^T\beta(t)^2dt=\infty. \tag{2.30}$$

Next, we take

$$\begin{cases} n=d, \quad \sigma(t)=I, \\ b(t,x)=\beta(t)\bar b, \quad r(t,x)=\beta(t), \quad \text{with } \bar b\in\mathbb{R}^n, \quad \bar b\ne\mathbf{1}. \end{cases} \tag{2.31}$$

Consequently, (2.26) implies

$$\theta(\cdot)=\beta(\cdot)[\bar b-\mathbf{1}],$$

which does not satisfy (2.29). Thus, (iv) is strictly weaker than (i).

We recall that under (A1), (A3) and (2.29), the following gives an equivalent martingale measure:

$$\mathbf{Q}(B)=\int_B e^{-\frac{1}{2}\int_0^T|\theta(s)|^2ds-\int_0^T\langle\theta(s),dW(s)\rangle}d\mathbf{P}, \quad \forall B\in\mathcal{F}_T. \tag{2.32}$$

Let us return to Problem (C). From Proposition 2.1, we know that $\mathcal{Y}_0$ contains $\mathcal{D}(g)\bigcap[0,\infty)$. We now would like to give a more precise characterization of $\mathcal{Y}_0$.

PROPOSITION 2.3. *Let (A1)–(A3) hold. Let* $\mathbf{Q}$ *be defined by (2.32) and let*

$$\eta_0=E_{\mathbf{Q}}\Big\{e^{-\int_0^T r(\tau,X(\tau))d\tau}y_0\Big\}, \tag{2.33}$$

Then

$$\mathcal{Y}_0 \subseteq \begin{cases} (\eta_0, \infty), & \text{if} \quad y_0 \notin \mathcal{D}(g), \\ [\eta_0, \infty), & \text{if} \quad y_0 \in \mathcal{D}(g). \end{cases} \tag{2.34}$$

If (A1)′, (A2)–(A3) hold, then the equality in (2.34) holds.

Proof. First, we assume that $y_0 \notin \mathcal{D}(g)$. For any $y \in \mathcal{Y}_0$, by definition, there exists a $\pi(\cdot) \in \Pi[0,T]$ such that $Y(T; y, \pi(\cdot)) \in \mathcal{D}(g)$. We recall the Girsanov transformation ([2,11]). Define $\mathbf{Q}$ as in (2.32). Then $\mathbf{Q}$ is a probability on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$, and it is equivalent to $\mathbf{P}$. Moreover, the following process

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s) ds, \quad t \in [0,T] \tag{2.35}$$

is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$, still with $\{\mathcal{F}_t\}_{t\geq 0}$ being the natural filtration. Now, (2.27) becomes

$$\begin{cases} dY(t) = r(t, X(t))Y(t)dt + \langle \sigma(t, X(t))^T \pi(t), d\widetilde{W}(t)\rangle, \quad t \in [0,T], \\ Y(0) = y. \end{cases} \tag{2.36}$$

Consequently, we must have

$$\begin{aligned} Y(T; y, \pi(\cdot)) &= e^{\int_0^T r(\tau, X(\tau))d\tau}\big[y + \textstyle\int_0^T e^{-\int_0^t r(\tau, X(\tau))d\tau} \langle \sigma(t, X(t))^T \pi(t), d\widetilde{W}(t)\rangle\big] \\ &> y_0. \end{aligned} \tag{2.37}$$

This leads to

$$y + \int_0^T e^{-\int_0^t r(\tau, X(\tau))d\tau} \langle \sigma(t, X(t))^T \pi(t), d\widetilde{W}(t)\rangle > e^{-\int_0^T r(\tau, X(\tau))d\tau} y_0. \tag{2.38}$$

Taking expectation with respect to $\mathbf{Q}$, we obtain

$$y > E_{\mathbf{Q}}\Big\{ e^{-\int_0^T r(\tau, X(\tau))d\tau} y_0 \Big\} \equiv \eta_0. \tag{2.39}$$

This implies

$$\mathcal{Y}_0 \subseteq (\eta_0, \infty). \tag{2.40}$$

In the case that $y_0 \in \mathcal{D}(g)$ we may simply replace "$>$" in (2.37)–(2.39) by "$\geq$". Thus, (2.34) holds.

Now, let us assume (A1)′, (A2)–(A3). For any real number $\widetilde{y}_0 > y_0$, we consider the following:

$$\begin{cases} d\widetilde{Y}(t) &= \{r(t,X(t))\widetilde{Y}(t) + \langle h(t,X(t)),\widetilde{\pi}(t)\rangle\}dt \\ &\quad +\langle\sigma(t,X(t))^T\widetilde{\pi}(t), dW(t)\rangle, \quad t\in[0,T], \\ \widetilde{Y}(T) &= \widetilde{y}_0. \end{cases} \tag{2.41}$$

This equation is called a backward stochastic differential equation (BSDE, for short). Next, we apply an idea from Yong (1999) (see Ma and Yong (1999) and Yong (2000) also) to solve the above BSDE. Suppose $(\widetilde{Y}(\cdot),\widetilde{\pi}(\cdot))$ is an adapted solution of (2.41) which admits a representation of the following form:

$$\widetilde{Y}(t) = \widetilde{v}(t,X(t)), \quad t\in[0,T],\ a.s., \tag{2.42}$$

for some smooth function $\widetilde{v}(\cdot\,,\cdot)$. By Itô's formula, we have

$$\begin{aligned} d\widetilde{Y}(t) &= \{\widetilde{v}_t(t,X(t)) + \langle\widetilde{v}_x(t,X(t)), b(t,X(t))\rangle \\ &\quad +\tfrac{1}{2}\mathrm{tr}\big[\widetilde{v}_{xx}(t,X(t))\sigma(t,X(t))\sigma(t,X(t))^T\big]\}dt \\ &\quad +\langle\widetilde{v}_x(t,X(t)),\sigma(t,X(t))dW(t)\rangle. \end{aligned} \tag{2.43}$$

Comparing (2.43) with (2.41), we see that $\widetilde{v}(\cdot\,,\cdot)$ should be chosen such that

$$\begin{cases} \sigma(t,X(t))^T[\widetilde{\pi}(t) - \widetilde{v}_x(t,X(t))] = 0, \\ \widetilde{v}_t(t,X(t)) + \langle\widetilde{v}_x(t,X(t)), b(t,X(t))\rangle + \frac{1}{2}\mathrm{tr}\big[\widetilde{v}_{xx}(t,X(t))\sigma(t,X(t))\sigma(t,X(t))^T\big] \\ = r(t,X(t)\widetilde{v}(t,X(t)) + \langle h(t,X(t)),\widetilde{\pi}(t)\rangle. \end{cases}$$

Thus, one natural choice is that

$$\widetilde{\pi}(t) = \widetilde{v}_x(t,X(t)), \quad t\in[0,T],\ a.s., \tag{2.44}$$

with $\widetilde{v}(\cdot\,,\cdot)$ being a solution of the following linear parabolic partial differential equation:

$$\begin{cases} \widetilde{v}_t + \frac{1}{2}\mathrm{tr}\big[\sigma\sigma^T\widetilde{v}_{xx}\big] + r\langle\mathbf{1},\widetilde{v}_x\rangle - r\widetilde{v} = 0, \\ \widetilde{v}(T,x) = \widetilde{y}_0. \end{cases} \tag{2.45}$$

By Yong (1999), we know that when r,σ are smooth, (2.45) admits a unique classical solution $\widetilde{v}$, and if $X(\cdot)$ is the solution of (1.3), then we can show that (2.42) and (2.44) give an adapted solution to (2.41). This means $\widetilde{Y}(\cdot) = Y(\cdot;\widetilde{Y}(0),\widetilde{\pi}(\cdot))$. Then similar to the proof of Proposition 2.2, we obtain

$$\begin{aligned} \widetilde{y}_0 &= \widetilde{Y}(T) = Y(T;\widetilde{Y}(0),\widetilde{\pi}(\cdot)) \\ &= e^{\int_0^T r(\tau,X(\tau))d\tau}\{\widetilde{Y}(0) + \int_0^T e^{-\int_0^t r(\tau,X(\tau))d\tau}\langle\sigma(t,X(t))^T\widetilde{\pi}(t), d\widetilde{W}(t)\rangle\}. \end{aligned} \tag{2.46}$$

Thus,

$$\widetilde{Y}(0) = E_{\mathbf{Q}}\Big\{e^{-\int_0^T r(\tau,X(\tau))d\tau}\widetilde{y}_0\Big\} \in \mathcal{Y}_0, \quad \forall \widetilde{y}_0 > y_0. \tag{2.47}$$

Since $\widetilde{y}_0 \mapsto E_{\mathbf{Q}}\Big\{e^{-\int_0^T r(\tau,X(\tau))d\tau}\widetilde{y}_0\Big\}$ is strictly increasing, we see that for any $y > \eta_0$, there exists a $\widetilde{y}_0 > y_0$ such that

$$y = E_{\mathbf{Q}}\Big\{e^{-\int_0^T r(\tau,X(\tau))d\tau}\widetilde{y}_0\Big\} \in \mathcal{Y}_0. \tag{2.48}$$

This, together with (2.34), leads to

$$\mathcal{Y}_0 = (\eta_0, \infty). \tag{2.49}$$

The case $y_0 \in \mathcal{D}(g)$ can be proved similarly. ∎

We have seen that (A3) allow us to get equality in (2.34) for $\mathcal{Y}_0$. Also, Proposition 2.2 tells us that (A3) is almost necessary for $\mathcal{Y}_f \neq \phi$. However, we do not see a direct proof for the sufficiency of $\mathcal{Y}_f \neq \phi$ from (A1)–(A3).

3. EXISTENCE OF OPTIMAL PORTFOLIOS

It is known that Pontryagin's *maximum principle* is one of powerful tools in optimal control theory (Yong and Zou (1999)). In this section, we are going to use the idea of maximum principle to establish the existence of optimal portfolios for Problem (C), which leads to $\mathcal{Y}_s \neq \phi$. To this end, let (A1)–(A3) hold, $y \in \mathcal{Y}_s$, and let $\overline{\pi}(\cdot) \in \Pi[0,T]$ be an optimal portfolio of Problem (C), whose corresponding optimal wealth process is denoted by $\overline{Y}(\cdot) \triangleq Y(\cdot,;y,\overline{\pi}(\cdot))$. Now, let $\pi(\cdot) \in \Pi[0,T]$ such that $J(y;\pi(\cdot))$ is finite. For any $\delta \in (0,1)$, denote

$$\pi_\delta(\cdot) = \overline{\pi}(\cdot) + \delta[\pi(\cdot) - \overline{\pi}(\cdot)]. \tag{3.1}$$

By the concavity of $g(\cdot)$, we have

$$\begin{aligned} -\infty &< (1-\delta)J(y;\overline{\pi}(\cdot)) + \delta J(y;\pi(\cdot)) \\ &\leq J(y;\pi_\delta(\cdot)) \leq \sup_{\pi(\cdot)\in\Pi[0,T]} J(y;\pi(\cdot)) < \infty, \quad \forall \delta \in (0,1). \end{aligned} \tag{3.2}$$

By the optimality of $\overline{\pi}(\cdot)$ and (3.2), we have (denoting $Y_\delta(\cdot) \triangleq Y(\cdot\,;y,\pi_\delta(\cdot))$ to be the wealth process corresponding to $\pi_\delta(\cdot)$)

$$\begin{aligned} 0 &\geq \tfrac{1}{\delta}\big\{J(y;\pi_\delta(\cdot)) - J(y;\overline{\pi}(\cdot))\big\} = \tfrac{1}{\delta}E\big\{g(Y_\delta(T)) - g(\overline{Y}(T))\big\} \\ &\geq J(y;\pi(\cdot)) - J(y;\overline{\pi}(\cdot)) = E\big\{g(Y(T;y,\pi(\cdot))) - g(\overline{Y}(T))\big\}. \end{aligned} \tag{3.3}$$

Sending $\delta \to 0$ in the above yields

$$0 \geq E\big\{g'(\overline{Y}(T))\eta(T)\big\} \geq E\big\{g(Y(T;y,\pi(\cdot))) - g(\overline{Y}(T))\big\}, \tag{3.4}$$

where $\eta(\cdot)$ is the solution of the following *variational system*:

$$\begin{cases} d\eta(t) &= \big\{r(t,X(t))\eta(t) + \langle\theta(t), \sigma(t,X(t))^T[\pi(t) - \overline{\pi}(t)]\rangle\big\}dt \\ &\quad +\langle\sigma(t,X(t))^T[\pi(t)-\overline{\pi}(t)], dW(t)\rangle, \quad t \in [0,T], \\ \eta(0) &= 0, \end{cases} \tag{3.5}$$

and for the time being, we assume that

$$g'(\overline{Y}(T)) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}). \tag{3.6}$$

Next, we introduce the following *adjoint equation* of (3.5):

$$\begin{cases} d\varphi(t) = -r(t,X(t))\varphi(t)dt + \langle\psi(t), dW(t)\rangle, \quad t\in[0,T], \\ \varphi(T) = g'(\overline{Y}(T)). \end{cases} \tag{3.7}$$

Again (3.7) is a BSDE. Let $(\varphi(\cdot),\psi(\cdot))$ be the adapted solution of (3.7). Using Itô's formula, we have

$$\begin{aligned} E\big\{g'(\overline{Y}(T))\eta(T)\big\} &= E\big\{\varphi(T)\eta(T)\big\} \\ &= E\int_0^T \langle\varphi(t)\theta(t) + \psi(t), \sigma(t,X(t))^T[\pi(t)-\overline{\pi}(t)]\rangle dt. \end{aligned} \tag{3.8}$$

Combining (3.8) and (3.4), one obtains

$$\begin{aligned} 0 \ &\geq E\int_0^T \langle\sigma(t,X(t))\big[\varphi(t)\theta(t)+\psi(t)\big], \pi(t)-\overline{\pi}(t)\rangle dt, \\ &\qquad \forall \pi(\cdot)\in\Pi[0,T], \text{such that } J(y;\pi(\cdot)) \text{ is finite.} \end{aligned} \tag{3.9}$$

Conversely, suppose $\overline{\pi}(\cdot)\in\Pi[0,T]$ such that (3.9) holds with $(\varphi(\cdot),\psi(\cdot))$ being the adapted solution of (3.7), and $\overline{Y}(\cdot)$ being the corresponding solution of (1.5). Let $\pi(\cdot)\in\Pi[0,T]$ such that $J(y;\pi(\cdot))$ is finite. Then, by Taylor expansion, (3.8)–(3.9), and the convexity of $g(\cdot)$, we have

$$\begin{aligned} &J(y;\pi(\cdot)) - J(y;\overline{\pi}(\cdot)) \\ &= E\Big\{\int_0^T \langle\varphi(t)\theta(t)+\psi(t), \sigma(t,X(t))^T[\pi(t)-\overline{\pi}(t)]\rangle dt \\ &\qquad\qquad + \tfrac{1}{2}g''(\widetilde{Y})[Y(T)-\overline{Y}(T)]^2\Big\} \\ &\leq \tfrac{1}{2}E\big\{g''(\widetilde{Y})[Y(T)-\overline{Y}(T)]^2\big\} \leq 0. \end{aligned} \tag{3.10}$$

Thus, we obtain the following result.

THEOREM 3.1. *Let (A1)–(A3) hold. Let $\overline{\pi}(\cdot)\in\Pi[0,T]$ whose corresponding wealth process $\overline{Y}(\cdot)$ satisfies (3.6). Then $(\overline{Y}(\cdot),\overline{\pi}(\cdot))$ is an optimal* *pair for Problem (C) if and only if (3.9) holds with $(\varphi(\cdot),\psi(\cdot))$ being the adapted solution of BSDE (3.7).*

Now, let us note that

$$\begin{aligned}\mathcal{R}\big(\sigma(t,X(t))\big) &= \mathcal{N}\big(\sigma(t,X(t))^T\big)^{\perp}\\ &= \mathcal{N}\big(\sigma(t,X(t))\sigma(t,X(t))^T\big)^{\perp}\\ &= \mathcal{R}\big(\sigma(t,X(t))\sigma(t,X(t))^T\big).\end{aligned}\tag{3.11}$$

Thus, (2.13) is also equivalent to the existence of an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process $\hat{\theta}:[0,T]\times\Omega\to\mathbb{R}^n$ satisfying

$$h(t,X(t)) = \sigma(t,X(t))\sigma(t,X(t))^T\hat{\theta}(t),\quad \forall t\in[0,T],\ a.s.\tag{3.12}$$

We now introduce the following assumption which is a little stronger than (A3).

(A3)′ There exists a smooth function $\hat{\theta}:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ with bounded derivatives of all required orders, such that

$$h(t,x) = \sigma(t,x)\sigma(t,x)^T\hat{\theta}(t,x),\qquad \forall(t,x)\in[0,T]\times\mathbb{R}^n.\tag{3.13}$$

When (3.13) holds, equation (2.27) becomes

$$\begin{cases} dY(t) = \big\{r(t,X(t))Y(t) + \langle\sigma(t,X(t))^T\pi(t),\sigma(t,X(t))^T\hat{\theta}(t)\rangle\big\}dt\\ \qquad\qquad +\langle\sigma(t,X(t))^T\pi(t),dW(t)\rangle,\quad t\in[0,T],\\ Y(0)=y.\end{cases}\tag{3.14}$$

We are now in the position to state and prove the following existence theorem of optimal portfolios.

THEOREM 3.2. *Let (A1)′, (A2) and (A3)′ hold with either $y_0=0$, or $y_0=-\infty$. Then*

$$\mathcal{Y}_s=\mathcal{Y}_f=\mathcal{Y}_0=\mathcal{D}(g).\tag{3.15}$$

Consequently, for any initial endowment $y\in\mathcal{D}(g)$, there exists an optimal portfolio.

Proof. Note that in the cases $y_0=0$, or $y_0=-\infty$, one has $\eta_0=y_0$ (see (2.33) for the definition of η_0). Set

$$\theta(t)=\sigma(t,X(t))^T\hat{\theta}(t,X(t)),\quad \forall t\in[0,T],\ a.s.\tag{3.16}$$

Now we prove that for any $y \in (y_0, \infty)$, there exists a $\overline{\pi}(\cdot) \in \Pi[0,T]$ such that (3.6) holds and

$$\varphi(t)\theta(t) + \psi(t) = 0, \quad t \in [0,T], \ a.s., \tag{3.17}$$

where $(\varphi(\cdot), \psi(\cdot))$ is the adapted solution of (3.7). Then (3.9) holds and $\overline{\pi}(\cdot)$ will be an optimal portfolio, which implies $y \in \mathcal{Y}_s$.

Note that if (3.17) holds, then $\varphi(\cdot)$ should satisfy the following:

$$d\varphi(t) = -\varphi(t)\big\{r(t, X(t))dt + \langle \theta(t), dW(t)\rangle\big\}, \qquad t \in [0,T]. \tag{3.18}$$

This implies

$$\varphi(t) = \varphi(0)e^{-\int_0^t [r(\tau, X(\tau)) + \frac{1}{2}|\theta(\tau)|^2]d\tau + \int_0^t \langle \theta(\tau), dW(\tau)\rangle} \triangleq \varphi(0)\rho(t), \quad t \in [0,T], \tag{3.19}$$

where

$$\rho(t) = e^{-\int_0^t [r(\tau, X(\tau)) + \frac{1}{2}|\theta(\tau)|^2]d\tau + \int_0^t \langle \theta(\tau), dW(\tau)\rangle} \quad t \in [0,T], \tag{3.20}$$

and $\varphi(0) \in \mathbb{R}$. By a simple calculation, using Itô's formula, we have

$$E|\rho(T)|^\alpha \le C_\alpha, \qquad \forall \alpha \in \mathbb{R}. \tag{3.21}$$

We want to choose a proper $\varphi(0) \in \mathbb{R}$ so that

$$g'(\overline{Y}(T)) = \varphi(T) \equiv \varphi(0)\rho(T). \tag{3.22}$$

By (A2), $g'(\cdot)$ is strictly decreasing, and with the range $(0,\infty)$. Thus, (3.22) is equivalent to the following:

$$\overline{Y}(T) = (g')^{-1}\big(\varphi(0)\rho(T)\big). \tag{3.23}$$

Using (2.6) and (3.21), we obtain that

$$E\Big|(g')^{-1}(\varphi(0)\rho(T))\Big|^2 \le C, \tag{3.24}$$

with the constant only depending on $|\varphi(0)|$. Now, we consider the following BSDE:

$$\begin{cases} d\overline{Y}(t) = \big\{r(t, X(t))\overline{Y}(t) + \langle h(t, X(t)), \overline{\pi}(t)\rangle\big\}dt \\ \qquad\qquad\qquad + \langle \sigma(t, X(t))^T \overline{\pi}(t), dW(t)\rangle, \quad t \in [0,T], \\ \overline{Y}(T) = (g')^{-1}\big(\varphi(T)\big). \end{cases} \tag{3.25}$$

Note that in the above BSDE, the terminal value $\overline{Y}(T)$ is not a constant. Thus, the technique used in the proof of Proposition 2.6 has to be modified.

To this end, we introduce $\xi(t) = \ln \varphi(t)$. Then $\xi(\cdot)$ satisfies the following SDE:

$$\begin{aligned} d\xi(t) = -\big\{ r(t, X(t)) + \tfrac{1}{2} |\sigma(t, X(t))^T \hat\theta(t, X(t))|^2 \big\} dt \\ + \langle \hat\theta(t, X(t)), \sigma(t, X(t)) dW(t) \rangle, \quad t \in [0, T]. \end{aligned} \tag{3.26}$$

Consequently, we have a decoupled FBSDE (1.3), (3.26) and (3.25). Now, suppose $(X(\cdot), \xi(\cdot), \overline{Y}(\cdot), \overline{\pi}(\cdot))$ is an adapted solution to this FBSDE such that

$$\overline{Y}(t) = u(t, X(t), \xi(t)), \quad t \in [0, T], \ a.s., \tag{3.27}$$

for some smooth function $u(t, x, \xi)$. By Itô's formula, (and suppressing t and $X(t)$) we have

$$\begin{aligned} d\overline{Y} \ &= \big\{ u_t + \langle u_x, b \rangle - u_\xi [r + \tfrac{1}{2} |\sigma^T \hat\theta|^2 \\ &\quad + \tfrac{1}{2} \mathrm{tr}\big[(u_{xx} + 2 u_{x\xi} \hat\theta^T + u_{\xi\xi} \hat\theta \hat\theta^T) \sigma \sigma^T \big] \big\} dt \\ &\quad + \langle u_x + u_\xi \hat\theta, \sigma dW(t) \rangle. \end{aligned} \tag{3.28}$$

Comparing (3.28) with (3.25), we see that a proper choice for $u(\cdot)$ should lead to

$$\overline{\pi}(t) = u_x(t, X(t), \xi(t)) + u_\xi(t, X(t), \xi(t)) \hat\theta(t, X(t)), \quad t \in [0, T], \tag{3.29}$$

and the following PDE is satisfied:

$$\begin{cases} u_t + \tfrac{1}{2} \mathrm{tr}\big[(u_{xx} + 2 u_{x\xi} \hat\theta^T + u_{\xi\xi} \hat\theta \hat\theta^T) \sigma \sigma^T \big] \\ \qquad + \langle u_x, \mathbf{1} \rangle r - u_\xi [r + \tfrac{1}{2} |\sigma^T \hat\theta|^2 + \langle h, \hat\theta \rangle] - r u = 0, \\ u(T, x, \xi) = (g')^{-1}(e^\xi). \end{cases} \tag{3.30}$$

Similar to [9], we can find a classical solution $u(\cdot)$ to (3.30). Then (3.27) and (3.29) gives an adapted solution $(\overline{Y}(\cdot), \overline{\pi}(\cdot))$ to BSDE (3.25).

Next, we want to choose a suitable $\varphi(0) \in \mathbb{R}$ such that the following is true:

$$\overline{Y}(0) = y. \tag{3.31}$$

Recalling (2.5), and using comparison theorem of BSDEs (see [1]), we can prove that in either case $y_0 = 0$, or $y_0 = -\infty$,

$$\begin{cases} \text{as } \varphi(0) \to 0, \quad (g')^{-1}\big(\varphi(0)\rho(T)\big) \to \infty, \ \& \ \overline{Y}(0) \to +\infty, \\ \text{as } \varphi(0) \to +\infty, \quad (g')^{-1}\big(\varphi(0)\rho(T)\big) \to y_0, \ \& \ \overline{Y}(0) \to y_0. \end{cases} \tag{3.32}$$

Hence, by the continuity of $\varphi(0) \mapsto \overline{Y}(0)$, in either case, there exists a $\varphi(0) \in (y_0, \infty)$ such that (3.31) holds. This means that for $y \in (\eta_0, \infty)$, there exist $(\overline{Y}(\cdot), \overline{\pi}(\cdot))$ and $(\varphi(\cdot), \psi(\cdot))$ satisfying (2.27), (3.7) (since (3.17) holds), and (3.9). Thus, by Theorem 3.1, the there exists an optimal portfolio for y. This proves our conclusion for the case $y_0 \notin \mathcal{D}(g)$. Finally, in the case $y_0 = 0 \in \mathcal{D}(g)$, by (2.9) and (2.34), we need only to show that $0 \in \mathcal{Y}_s$. By (A3)$'$, we know that there is no-arbitrage for the market. Thus, the optimal portfolio for $y = 0$ is $\overline{\pi}(\cdot) = 0$. This gives the solvability of Problem (C) at $y = 0$. ▮

4. CONSTRUCTION OF AN OPTIMAL PORTFOLIO

In this section, we concentrate on a special case for which we can construct an optimal portfolio via a Riccati type equation. We know that the Riccati equation was usually used in some linear quadratic optimal control problems (see Chen and Yong (2000) and Yong (2000) for some details). Here, we use it to treat some problem which is not linear-quadratic.

In what follows, we assume that (A1)$'$ and (A3)$'$ hold. Let $\gamma \in (0,1)$ and define

$$g(y) = \begin{cases} \frac{1}{\gamma} y^\gamma, & y \geq 0, \\ -\infty, & y < 0. \end{cases} \tag{4.1}$$

Let us now present a formal derivation. Consider the following BSDE:

$$\begin{cases} dp(t) = \eta(t)dt + \langle \sigma(t, X(t))^T \zeta(t), dW(t) \rangle, & t \in [0,T], \\ p(T) = 1, \end{cases} \tag{4.2}$$

where $\eta(\cdot)$ is undetermined. Since $\sigma(t, X(t))$ is not necessarily invertible, the existence of an adapted solution is not obvious. We will address this problem a little later. Now, suppose $(p(\cdot), \zeta(\cdot))$ is an adapted solution of (4.2) such that $p(t) > 0$ for all $t \in [0,T]$, almost surely. Next, for given $\pi(\cdot) \in \Pi[0,T]$ and $y \in [0, \infty)$, let $Y(\cdot)$ be the corresponding solution to (3.14). By Itô's formula, we obtain that (we suppress t and $X(t)$ in the following)

$$\begin{aligned} d[Y^\gamma] \ &= \big\{ \gamma Y^{\gamma-1}(rY + \langle \sigma^T \hat{\theta}, \sigma^T \pi \rangle) + \tfrac{\gamma(\gamma-1)}{2} Y^{\gamma-2} |\sigma^T \pi|^2 \big\} dt \\ &\qquad + \gamma Y^{\gamma-1} \langle \sigma^T \pi, dW(t) \rangle. \end{aligned} \tag{4.3}$$

Hence,

$$\begin{aligned} d[pY^\gamma] \ &= \Big\{ \eta Y^\gamma + p\big[\gamma Y^{\gamma-1}(rY + \langle \sigma^T \hat{\theta}, \sigma^T \pi \rangle) + \tfrac{\gamma(\gamma-1)}{2} Y^{\gamma-2} |\sigma^T \pi|^2 \big] \\ &\qquad + \gamma Y^{\gamma-1} \langle \sigma^T \zeta, \sigma^T \pi \rangle \Big\} dt + [\cdots] dW(t). \end{aligned} \tag{4.4}$$

This leads to

$$\begin{aligned} J(y;\pi(\cdot)) &\equiv \tfrac{1}{\gamma}EY(T)^\gamma = \tfrac{1}{\gamma}E[p(T)Y(T)^\gamma] \\ &= \tfrac{1}{\gamma}p(0)y^\gamma + E\int_0^T \Big\{\tfrac{\eta Y^\gamma}{\gamma} + p\big[Y^{\gamma-1}(rY + \langle\sigma^T\hat\theta,\sigma^T\pi\rangle) \\ &\quad + \tfrac{\gamma-1}{2}Y^{\gamma-2}|\sigma^T\pi|^2\big] + Y^{\gamma-1}\langle\sigma^T\zeta,\sigma^T\pi\rangle\Big\}dt \\ &= \tfrac{1}{\gamma}p(0)y^\gamma + E\int_0^T\Big\{\Big[\tfrac{\eta}{\gamma} + rp + p\tfrac{|\sigma^T(\hat\theta+\frac{\zeta}{p})|^2}{2(1-\gamma)}\Big]Y^\gamma \\ &\quad - \tfrac{1-\gamma}{2}pY^{\gamma-2}\Big|\sigma^T\big[\pi - \tfrac{\hat\theta+\frac{\zeta}{p}}{1-\gamma}Y\big]\Big|^2\Big\}dt. \end{aligned} \tag{4.5}$$

Set

$$\hat\zeta(t) = \frac{\zeta(t)}{p(t)}, \quad t\in[0,T], \tag{4.6}$$

and choose

$$\eta = -\gamma p\big[r + \frac{|\sigma^T(\hat\theta+\hat\zeta)|^2}{2(1-\gamma)}\big]. \tag{4.7}$$

Then $(p,\hat\zeta)$ is an adapted solution of the following BSDE:

$$\begin{cases} dp = p\Big\{-\gamma\big[r + \frac{|\sigma^T(\hat\theta+\hat\zeta)|^2}{2(1-\gamma)}\big]dt + \langle\sigma^T\hat\zeta, dW(t)\rangle\Big\}, \quad t\in[0,T], \\ p(T) = 1. \end{cases} \tag{4.8}$$

This is equivalent to the following (noting (4.6))

$$\begin{cases} dp = -\gamma\big[rp + \frac{|\sigma^T(p\hat\theta+\zeta)|^2}{2(1-\gamma)p}\big]dt + \langle\sigma^T\zeta, dW(t)\rangle, \quad t\in[0,T], \\ p(T) = 1. \end{cases} \tag{4.9}$$

This is called a *stochastic Riccati equation*. One can find a similar thing in stochastic linear quadratic optimal control problems (Chen and Yong (2000) and Yong (2000)).

Further, we set $\hat p = \log p$. By Itô's formula, one sees that $(\hat p,\hat\zeta)$ is an adapted solution of the following BSDE:

$$\begin{cases} d\hat p = -\big[r\gamma + \frac{\gamma|\sigma^T\hat\theta|^2}{2(1-\gamma)} + \frac{\gamma}{1-\gamma}\langle\sigma^T\hat\theta,\sigma^T\hat\zeta\rangle + \frac{1}{2(1-\gamma)}|\sigma^T\hat\zeta|^2\big]dt \\ \qquad\quad + \langle\sigma^T\hat\zeta, dW(t)\rangle, \quad t\in[0,T], \\ \hat p(T) = 0. \end{cases} \tag{4.10}$$

With the choice η given by (4.7), it follows from (4.5) that

$$\begin{aligned} J(y;\pi(\cdot)) &\equiv \tfrac{1}{\gamma}EY(T)^\gamma \\ &= \tfrac{1}{\gamma}p(0)y^\gamma - \tfrac{1-\gamma}{2}\int_0^T pY^{\gamma-2}\Big|\sigma^T\big[\pi - \tfrac{\hat\theta+\hat\zeta}{1-\gamma}Y\big]\Big|^2 dt. \end{aligned} \tag{4.11}$$

Note that $p(\cdot)$ is independent of $\pi(\cdot)$. Thus, if we denote

$$\overline{\pi}(t)=\frac{\hat{\theta}(t,X(t))+\hat{\zeta}(t)}{1-\gamma}\overline{Y}(t),\quad t\in[0,T], \tag{4.12}$$

with $\overline{Y}(\cdot)$ being the corresponding solution of (3.14), and if such a $\overline{\pi}(\cdot)\in\Pi[0,T]$, then

$$J(y;\pi(\cdot))\le J(y;\overline{\pi}(\cdot))=\frac{1}{\gamma}p(0)y^{\gamma},\quad \forall\pi(\cdot)\in\Pi[0,T]. \tag{4.13}$$

This means that the $\overline{\pi}(\cdot)$ defined by (4.12) is an optimal portfolio. Plugging this portfolio into (3.14), we have

$$\begin{cases} d\overline{Y}(t)=\overline{Y}(t)\big\{r(t,X(t))+\frac{\langle\sigma(t,X(t))^T[\hat{\theta}(t)+\hat{\zeta}(t)],\sigma(t,X(t))^T\hat{\theta}(t)\rangle}{1-\gamma}\big\}dt \\ \qquad\qquad +\frac{\overline{Y}(t)}{1-\gamma}\langle\sigma(t,X(t))^T[\hat{\theta}(t)+\hat{\zeta}(t)],dW(t)\rangle,\qquad t\in[0,T], \\ \overline{Y}(0)=y. \end{cases} \tag{4.14}$$

Consequently, by Itô's formula, we have

$$\begin{aligned} d[\ln\overline{Y}(t)]=\Big\{&r(t,X(t))+\frac{|\sigma(t,X(t))^T[\gamma\hat{\theta}(t,X(t))+\hat{\zeta}(t)]|^2}{2(1-\gamma)^2} \\ &-\frac{1}{2}|\sigma(t,X(t))^T\hat{\theta}(t,X(t))|^2\Big\}dt \\ &+\langle\frac{\sigma(t,X(t))^T[\hat{\theta}(t,X(t))+\hat{\zeta}(t)]}{1-\gamma},dW(t)\rangle, \end{aligned} \tag{4.15}$$

Thus, the following holds true:

$$\begin{aligned} \overline{Y}(t)\ &=ye^{\int_0^t\big\{r(s,X(s))+\frac{|\sigma(s,X(s))^T[\gamma\hat{\theta}(s,X(s))+\hat{\zeta}(s)]|^2}{2(1-\gamma)^2}-\frac{1}{2}|\sigma(s,X(s))^T\hat{\theta}(s,X(s))|^2\big\}ds} \\ &\quad\cdot e^{\int_0^t\langle\frac{\sigma(s,X(s))^T[\hat{\theta}(s,X(s))+\hat{\zeta}(s)]}{1-\gamma},dW(s)\rangle},\qquad t\in[0,T]. \end{aligned} \tag{4.16}$$

Combining (4.12), we have the expression for $\overline{\pi}(\cdot)$. Under certain conditions, this $\overline{\pi}(\cdot)$ will be feasible. Then we can further claim that it is optimal.

To summarize the above, we state the following proposition.

PROPOSITION 4.1. *Let (A1)′ and (A3)′ hold. Suppose that BSDE (4.8) admits an adapted solution* $(p(\cdot),\hat{\zeta}(\cdot))$ *such that* $\overline{\pi}(\cdot)$ *defined by (4.12) with* $\overline{Y}(\cdot)$ *defined by (4.16) is feasible. Then* $\overline{\pi}(\cdot)$ *is an optimal portfolio and* $\overline{Y}(\cdot)$ *is the corresponding optimal wealth process.*

The remaining problem is to make investigate when BSDE (4.8) admits an adapted solution $(p(\cdot),\hat{\zeta}(\cdot))$ such that the conditions of Proposition 4.1

hold. Note that when $n = d = 1$ with $\sigma(t, X(t))^{-1}$ bounded, by Lepeltier and Martin (1998), we know that BSDE (4.10) admits a maximal adapted solution $(\hat{p}, \hat{\zeta})$ with $\hat{p} \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ and $\hat{\zeta} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$, which will give the existence of an adapted solution for (4.8). However, since we do not have assumptions on n and d, as well as the rank of the matrix $\sigma(t, X(t))$, the technique of Kramakov and Schachermayer (1999) does not apply here.

We now apply the idea of "Four-Step-Scheme" again (Ma and Yong (1999)). Let us assume that $(p(\cdot), \hat{\zeta}(\cdot))$ is an adapted solution of (4.8), and it has the following form:

$$p(t) = v(t, X(t)), \quad t \in [0, T], \tag{4.17}$$

with $v(\cdot\,, \cdot)$ being some smooth function. It is reasonable to assume this because all the randomness of the BSDE (4.8) come from $X(\cdot)$ (and from $W(\cdot)$, of course). Applying Itô's formula, we have (suppressing $(t, X(t))$ below)

$$\begin{aligned} &\big\{v_t + \langle b, v_x\rangle + \tfrac{1}{2}\mathrm{tr}[\sigma\sigma^T v_{xx}]\big\}dt + \langle v_x, \sigma dW(t)\rangle \\ &= dp = -v\gamma\big[r + \tfrac{|\sigma^T(\hat{\theta}+\hat{\zeta})|^2}{2(1-\gamma)}\big]dt + v\langle\sigma^T\hat{\zeta}, dW(t)\rangle. \end{aligned} \tag{4.18}$$

Comparing the diffusion terms on the two sides, we should have

$$\sigma(t, X(t))^T[v_x(t, X(t)) - v(t, X(t))\hat{\zeta}(t)] = 0, \qquad t \in [0, T], \ a.s. \tag{4.19}$$

Therefore, we had better choose

$$\hat{\zeta}(t) = \frac{v_x(t, X(t))}{v(t, X(t))}, \qquad t \in [0, T], \ a.s. \tag{4.20}$$

Next, comparing the drift terms on the two sides of (4.18), and taking into account of (4.20), we see that $v(\cdot\,, \cdot)$ should satisfy

$$\begin{cases} v_t + \frac{1}{2}\mathrm{tr}[\sigma\sigma^T v_{xx}] + \langle b + \frac{\gamma}{1-\gamma}\sigma\sigma^T\hat{\theta}, v_x\rangle \\ \qquad + \big(\gamma r + \frac{\gamma|\sigma^T\hat{\theta}|^2}{2(1-\gamma)}\big)v + \frac{\gamma|\sigma^T v_x|^2}{2(1-\gamma)v} = 0, \qquad (t, x) \in [0, T] \times \mathbb{R}^n, \\ v\big|_{t=T} = 1. \end{cases} \tag{4.21}$$

According to the above analysis, we see that if (4.21) admits a classical solution v such that it is bounded itself together with its partial derivatives and $\frac{1}{v}$, then $(p(\cdot), \hat{\zeta}(\cdot))$ defined by (4.17) and (4.20) with $X(\cdot)$ be the solution of (1.3) is an adapted solution of (4.8). Moreover, conditions of Proposition 4.1 will be satisfied and $\overline{\pi}(\cdot)$ will be an optimal portfolio. Thus, everything is now reduced to solving equation (4.21).

The general solvability of the above equation is difficult since the nonlinear term involves a quadratic term in v_x, and the equation is degenerate. However, we have an interesting special case, which we now present.

We introduce the following further assumption.

(A1)″ Maps $r : [0,T] \to [0,\infty)$, $b : [0,T] \to \mathbb{R}^n$, $\sigma : [0,T] \to \mathbb{R}^{n\times d}$ are bounded.

In another word, under (A1)", all the coefficients are x-independent. In this case, (A3)′ should be replaced by the following:

(A3)″ There exists a bounded function $\hat\theta : [0,T] \to \mathbb{R}^n$ such that

$$h(t) \triangleq b(t) - r(t)\mathbf{1} = \sigma(t)\sigma(t)^T\hat\theta(t), \qquad \forall t \in [0,T]. \tag{4.22}$$

Now, we are looking for solution $v(\cdot)$ of (4.21) which is x-independent. This is possible since all the coefficients (including the terminal condition) are independent of x. Consequently, we need only solve an ordinary equation:

$$\begin{cases} v_t + \big(\gamma r + \frac{\gamma|\sigma^T\hat\theta|^2}{2(1-\gamma)}\big)v = 0, \qquad t \in [0,T], \\ v\big|_{t=T} = 1. \end{cases} \tag{4.23}$$

The solution is given by

$$v(t) = e^{\int_t^T \left(\gamma r(s) + \frac{\gamma|\sigma(s)^T\hat\theta(s)|^2}{2(1-\gamma)}\right)ds}, \quad t \in [0,T]. \tag{4.24}$$

In this case, we have

$$p(t) = v(t), \quad \hat\zeta(t) = 0, \qquad t \in [0,T], \ a.s. \tag{4.25}$$

Clearly, the conditions of Proposition 4.1 hold. Thus, $\overline{\pi}(\cdot)$ constructed via (4.12) is an optimal portfolio.

We have to admit that the general case is left widely open and we hope to come back to the problem in our future publications.

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